Chapter 23

Completeness

Introduction

In the previous two chapters, we have studied connectedness and compactness: in this chapter we discuss a third 'c', known as *completeness*. It comes from an issue with the definition of convergent sequences: recall that a sequence (x_n) in a metric space (X, d) is convergent if there is a point $x \in X$ for which $(d(x_n, x))$ is a null sequence, and the point x is the limit of the sequence. The problem is the following: in order to show that a given sequence is convergent, we need to identify the limit point x, and this can be difficult. So the question we seek to answer in this chapter is:

> Are there metric spaces for which we can identify convergent sequences without first having to identify their limit?

Our answer to this question turns out to be surprisingly useful: the spaces that we identify include many common metric spaces and the property that they have enables us to prove a powerful theorem, the Contraction Mapping Theorem, which is important in the solution of many problems of analysis, some of which we will consider. Indeed, in Chapter 24, *Fractals*, you will see that the Contraction Mapping Theorem plays an important role in the investigation of fractals.

Chapter guide

We first explain what we mean by a sequence that 'appears to converge'. We then identify those metric spaces for which such sequences *do* converge – we call these *complete* metric spaces.

We then discuss the Contraction Mapping Theorem – a powerful theorem that gives sufficient conditions to ensure that a continuous mapping of a certain type on a complete metric space must have a fixed point. The Contraction Mapping Theorem is a surprisingly useful result and we give several examples of its applications.

In the last section of the chapter, we show how to embed any metric space into a complete metric space.

1 Completeness

In analysis, we are frequently given a sequence in a metric space, and we wish to know whether it is convergent.

In Book D, Chapter 14, we defined a convergent sequence as follows:

Definition 1.1 *Definition 3.2 of Chapter 14*

Let (X, d) be a metric space. A sequence (a_k) in X **d-converges** to $a \in X$ if the sequence of real numbers $(d(a_k, a))$ is a null sequence.

When using this definition to show that a given sequence is convergent, we must first identify a possible limit of the sequence and then show that the sequence converges to it. In many instances, however, it is difficult to identify a possible limit. For example, consider the sequence of real numbers (a_n) given by

$$a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}.$$

See Book D, Chapter 13, Theorem 1.22.

The limit of this sequence is not obvious. Fortunately, we have the Monotone Convergence Theorem available, which states that a non-decreasing sequence of real numbers that is bounded above is convergent. This sequence is non-decreasing (since $a_{n+1} - a_n = \frac{1}{(n+1)^2} > 0$ for $n \in \mathbb{N}$) and is bounded above by 2 (since $a_n \leq 1 + \int_1^n \frac{1}{x^2}\,dx$). So, by the Monotone Convergence Theorem, we know that (a_n) converges even though we have not identified its limit.

The limit of this sequence is in fact $\frac{\pi^2}{6}$ – this was first calculated in the eighteenth century by Leonhard Euler.

Now consider the sequence (b_n) where

$$b_n = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^n}{n}.$$

In this case, $b_{n+1} - b_n = \frac{(-1)^{n+1}}{n+1}$, and so the sequence is neither monotonic increasing nor decreasing. However, it is possible to show the sequence is convergent.

Exercise 1.1

Let $\mathbb{R}$ have the Euclidean metric and define the sequence (b_n) by $b_n = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^n}{n}$.

(a) Show that the odd and even subsequences of (b_n), $(b_{2n-1})_{n\in\mathbb{N}}$ and $(b_{2n})_{n\in\mathbb{N}}$, are bounded monotonic sequences.

(b) Show that the sequence (d_n) given by $d_n = b_{2n-1} - b_{2n}$ is null.

(c) Hence show that the sequence (b_n) is convergent.

We have now seen two examples of convergent sequences in $\mathbb{R}$ for which we were able to prove convergence without needing to identify the limit. But, as we saw for the second sequence, proving convergence was fairly involved. What we would like to do is to find some property of convergent sequences that we should test for instead.

By considering a sequence such as $(1/n)$, which converges in $\mathbb{R}$ with the Euclidean metric but diverges in $(0, 1]$ with the (induced) Euclidean metric, we see that there will be an interaction between the property that

we identify and the metric space that we are working with. As a result, we identify three goals that are closely intertwined:

1. identify a property, P say, that all convergent sequences (in any metric space) possess
2. determining whether a given sequence has property P should not require identification of a potential limit for the sequence
3. for a large class of metric spaces, the set of sequences with property P is the same as the set of convergent sequences – that is, for these metric spaces, a sequence should be convergent if, and only if, it has property P. Ideally the class should include metric spaces that are of use to us, such as Euclidean spaces and $C[0,1]$ with the max metric.

In both of the example sequences (a_n) and (b_n) that we considered above, we made extensive use of the Monotone Convergence Theorem, and it would be reasonable to consider this as our starting point. Unfortunately, the Monotone Convergence Theorem (as its name suggests) makes extensive use of the fact that $\mathbb{R}$ can be ordered – by which we mean two real numbers can be compared: $x < y$, $x = y$ or $x > y$. This is a very strong property and not appropriate for generalisation here. (For example, it is hard to think of a natural way of ordering points in the plane that is compatible with our usual notion of distance.)

The property that we work with instead is based around a different idea: the distances between the terms of a convergent sequence. We discuss it in the next subsection.

1.1 Cauchy sequences

Suppose that (a_n) is a convergent sequence (in a metric space (X,d)) and let $a \in X$ be its limit. Then we can use the Triangle Inequality to deduce that for $n, m \in \mathbb{N}$

$$d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) = d(a_n, a) + d(a_m, a).$$

But $(d(a_n, a))$ and $(d(a_m, a))$ are (real) null sequences, and so by choosing n, m large enough, we can make $d(a_n, a_m)$ arbitrarily small. The following lemma states this observation more precisely.

Lemma 1.2

Let (X,d) be a metric space and let (a_n) be a convergent sequence in X. Then, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$d(a_n, a_m) < \varepsilon \text{ for each } n, m > N.$$

Proof Let (a_n) be a convergent sequence in X. Then there is an $a \in X$ such that

$$a_n \xrightarrow{d} a \text{ as } n \to \infty.$$

Let $\varepsilon > 0$. Then there is an $N \in \mathbb{N}$ such that, for each $n > N$,

$$d(a_n, a) < \tfrac{1}{2}\varepsilon.$$

We deduce from the Triangle Inequality that, for all $n, m > N$,

$$d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

as required. ■

The property of convergent sequences given in Lemma 1.2 is the one we seek.

Augustin-Louis Cauchy (1789–1857) was a prolific French mathematician whose name occurs in many areas of analysis. His book *Cours d'analyse* was one of the first to place the foundations of calculus on a rigorous basis.

Definition 1.3 *Cauchy sequence*

Let (X, d) be a metric space and let (a_n) be a sequence in X.

Then (a_n) is a **d-Cauchy sequence** if, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$d(a_n, a_m) < \varepsilon \text{ for each } n, m > N.$$

Remarks

1. When it is clear from the context, we suppress the d and just refer to a sequence (a_n) as being a Cauchy sequence.
2. Lemma 1.2 states that any convergent sequence in a metric space is a Cauchy sequence.
3. Thus, since they are convergent for the Euclidean metric, the sequences (a_n) given by $a_n = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{n^2}$ and (b_n) given by $b_n = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^n}{n}$ are $d^{(1)}$-Cauchy.
4. This definition does not require us to identify a limit of the sequence, or even to know whether the sequence *has* a limit or not.
5. We may assume that $m > n$ in the definition, since if $n = m$, then $d(a_n, a_m) = 0$ which is certainly smaller than ε, and if $m < n$, we can interchange n and m since $d(a_n, a_m) = d(a_m, a_n)$.
6. To show that a given sequence is *not* a Cauchy sequence, it is enough to find just *one* $\varepsilon > 0$ such that, for each $N \in \mathbb{N}$, there are n and m both larger than N with $d(a_n, a_m) \geq \varepsilon$.
7. If we wish to emphasise the metric, we say that (a_n) is a Cauchy sequence *for* d or *with respect to* d, or that (a_n) is a d-Cauchy sequence.

Exercise 1.2

Determine whether each of the following sequences is a d-Cauchy sequence in $\mathbb{R}$, where d denotes the Euclidean metric.

(a) $(1/n)$ (b) (n)

Exercise 1.3

Let (X, d) be a metric space and (a_n) an eventually constant sequence in X. Show that (a_n) is a d-Cauchy sequence.

We now derive some basic properties of Cauchy sequences.

Lemma 1.4

Let (X, d) be a metric space and let (a_n) be a d-Cauchy sequence in X. Then (a_n) is a bounded sequence in X – that is, there are an $a \in X$ and an $M > 0$ such that

$$\{a_n : n \in \mathbb{N}\} \subseteq B_d(a, M).$$

Proof Since (a_n) is a Cauchy sequence, there is an $N \in \mathbb{N}$ such that

$$d(a_n, a_m) < 1 \text{ for each } n, m > N.$$

Let $a = a_{N+1}$; then

$$d(a_n, a) < 1 \text{ whenever } n > N.$$

So, for each $n \in \mathbb{N}$,

$$d(a_n, a) \leq \max\{1, d(a_1, a), d(a_2, a), d(a_3, a), \ldots, d(a_N, a)\}.$$

This implies that

$$\{a_n : n \in \mathbb{N}\} \subseteq B_d(a, M),$$

where $M = 1 + \max\{1, d(a_1, a), d(a_2, a), d(a_3, a), \ldots, d(a_N, a)\}$. ■

Corollary 1.5

Each $d^{(1)}$-Cauchy sequence in $\mathbb{R}$ is bounded.

Proof Let (a_n) be a $d^{(1)}$-Cauchy sequence in $\mathbb{R}$. By Lemma 1.4, there are an $a \in \mathbb{R}$ and an $M > 0$ such that

$$|a_n - a| < M \text{ for each } n \in \mathbb{N}.$$

By the Triangle Inequality,

$$|a_n| = |a_n - a + a| \leq |a_n - a| + |a| < M + |a| \text{ for each } n \in \mathbb{N}.$$

So (a_n) is bounded. ■

We now show that if a subsequence of a Cauchy sequence converges, then the whole sequence must converge. This is not true for sequences in general – for example, the sequence $(1, -1, 1, -1, \ldots)$ does not converge (for the Euclidean metric) but has the convergent subsequences $(1, 1, 1, \ldots)$ and $(-1, -1, -1, \ldots)$.

Lemma 1.6

Let (X, d) be a metric space and let (a_n) be a Cauchy sequence in X. If (a_n) has a convergent subsequence with limit a in X, then (a_n) is convergent with limit a.

Proof Let (a_{n_k}) be a subsequence of (a_n) that d-converges to $a \in X$. We prove that (a_n) also d-converges to a.

Let $\varepsilon > 0$ be given. Since $a_{n_k} \to a$ as $k \to \infty$, there is a $K \in \mathbb{N}$ such that for each $k > K$,

$$d(a_{n_k}, a) < \tfrac{1}{2}\varepsilon.$$

Since (a_n) is a d-Cauchy sequence, there is an $N \in \mathbb{N}$ such that

$$d(a_n, a_m) < \tfrac{1}{2}\varepsilon \text{ for each } m, n > N.$$

Let $k > K$ be chosen so that $n_k > N$; then

$$d(a_n, a_{n_k}) < \tfrac{1}{2}\varepsilon \text{ for each } n > N.$$

Moreover, since $k > K$,

$$d(a_{n_k}, a) < \tfrac{1}{2}\varepsilon.$$

Hence by the Triangle Inequality for d, if $n > N$, then

$$d(a_n, a) \leq d(a_n, a_{n_k}) + d(a_{n_k}, a) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon \text{ for each } n > N.$$

Since $\varepsilon > 0$ is arbitrary, (a_n) converges to a in (X, d). ■

Remark

Thus for the sequence (b_n) given by $b_n = -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^n}{n}$, an alternative way to prove that it is convergent for the Euclidean metric is to show that (b_n) is a Cauchy sequence and then show that it has a convergent subsequence.

Exercise 1.4

Let (c_n) be the sequence of real numbers given by $c_n = \sum_{i=1}^{n} \frac{(-1)^i}{2^i}$.

(a) Show that (c_n) is a Cauchy sequence for the Euclidean metric.

(b) By considering the subsequence of even terms, show that (c_n) is a convergent sequence.

1.2 Cauchy sequences in $\mathbb{R}$

We have seen that in any metric space, each convergent sequence is a Cauchy sequence (Lemma 1.2). We now show that in $\mathbb{R}$ (with its usual metric) the Cauchy sequences are exactly the convergent sequences, thus providing further evidence that the Cauchy property is the one we seek. Since we already know from Lemma 1.2 that every convergent sequence in $\mathbb{R}$ is a Cauchy sequence, we need only now show that each Cauchy sequence is convergent in $\mathbb{R}$ for the Euclidean metric.

To do this, we recall from Corollary 1.5 that a Cauchy sequence in $\mathbb{R}$ is bounded, and then construct a monotonic subsequence of the Cauchy sequence. By the Monotone Convergence Theorem, this subsequence is convergent, and so by Lemma 1.6 the original Cauchy sequence is convergent. We need the following result that we proved in Chapter 22.

See Book D, Chapter 13, Theorem 1.22.

Lemma 1.7 *Lemma 1.1 of Chapter 22*

Let (a_n) be a sequence of real numbers. Then (a_n) contains a monotonic subsequence.

We are now in a position to show that every Cauchy sequence in $\mathbb{R}$ is convergent.

Theorem 1.8

Each $d^{(1)}$-Cauchy sequence in $\mathbb{R}$ is $d^{(1)}$-convergent.

Proof Let (a_n) be a Cauchy sequence of real numbers.

By Corollary 1.5, there is a positive real number M such that

$$|a_n| < M \text{ for each } n \in \mathbb{N}.$$

By Lemma 1.7, there exists a monotonic subsequence (a_{n_k}) of (a_n). Since (a_n) is a bounded sequence, the monotonic subsequence is also bounded. Hence, by the Monotone Convergence Theorem, the subsequence is convergent. Thus, by Lemma 1.6, (a_n) is convergent. ■

Combining this result with our earlier observation that convergent sequences in metric spaces are Cauchy sequences, we obtain the following result.

Corollary 1.9

A sequence in $\mathbb{R}$ is $d^{(1)}$-convergent if, and only if, it is a $d^{(1)}$-Cauchy sequence.

We have shown that the $d^{(1)}$-Cauchy sequences in $\mathbb{R}$ are precisely the $d^{(1)}$-convergent sequences. This enables us to determine whether a sequence in $\mathbb{R}$ is convergent without needing to identify the limit of the sequence – exactly what we wanted do. Moreover, we have shown that in any metric space, convergent sequences are Cauchy and so we have a candidate for our property P: being Cauchy. It remains to show that there are useful metric spaces for which being Cauchy is the same as being convergent.

In the next section, we show that many of our standard examples of metric spaces have exactly this property and we develop some methods for identifying metric spaces for which Cauchy sequences correspond to the convergent sequences.

2 Complete metric spaces

We have just seen that $\mathbb{R}$ with the usual Euclidean metric is an example of a metric space for which a sequence is convergent if, and only if, it is a Cauchy sequence. Our aim in this section is to develop techniques that enable us to identify other examples of spaces for which the Cauchy sequences are exactly the convergent sequences.

We start by giving a name to those metric spaces for which each Cauchy sequence is convergent.

Definition 2.1 *Complete metric space*

A metric space (X, d) is a **complete metric space** if each Cauchy sequence in (X, d) is convergent; otherwise it is an **incomplete metric space**.

Remarks

1. Note that the definition of complete requires convergence to a point of X. Exercise 2.1 emphasises this subtlety.
2. Theorem 1.8 tells us that $(\mathbb{R}, d^{(1)})$ is a complete metric space.

Exercise 2.1

Show that $(0, 1]$ with the induced Euclidean metric is incomplete.

Exercise 2.2

Let X be a set and let d_0 denote the discrete metric for X. Show that (X, d_0) is a complete metric space.

Thus we have at least two examples of complete metric spaces: $(\mathbb{R}, d^{(1)})$ and (X, d_0), for any set X. We now develop criteria that guarantee that a given space is complete and identify some other complete metric spaces.

2.1 Criteria for completeness

We start by investigating when subspaces of a metric space are complete. We have already seen that, although $(\mathbb{R}, d^{(1)})$ is complete, the subspace $(0, 1]$ with the induced Euclidean metric is incomplete since there are Cauchy sequences in $(0, 1]$ that do not converge within $(0, 1]$.

The sequence $(\frac{1}{n})$ is such a sequence.

Our next result shows that $(A, d_A^{(1)})$ is complete if A is any *closed* subset of $\mathbb{R}$. The proof relies on the following characterisation of the closure of a set in terms of closure points:

Worked Exercise 3.7 and Theorem 3.8 of Chapter 16

Let (X, d) be a metric space and $A \subseteq X$. Then

$$\mathrm{Cl}_{(X,d)}(A) = \{x \in X : x \text{ is a } d\text{-closure point of } A \text{ in } X\},$$

where $x \in X$ is a d-closure point of A if there is a sequence in A that d-converges to x.

The following proposition uses both the induced metric and the original metric, and care is required to ensure that the right metric is used at the right time.

Proposition 2.2

Let (X, d) be a metric space and let $A \subseteq X$. Let d_A denote the induced metric on A from d.

1. If (A, d_A) is a complete metric space, then A is d-closed.
2. If (X, d) is a complete metric space and A is d-closed, then (A, d_A) is a complete metric space.

Proof

1. *Let (A, d_A) be complete. We show that A is d-closed.*

   Since a set is d-closed if, and only if, it coincides with its d-closure (see Theorem 3.3 of Chapter 16), and since it is always the case that $A \subseteq \mathrm{Cl}_{(X,d)}(A)$, it is enough to show that $\mathrm{Cl}_{(X,d)}(A) \subseteq A$. So let $x \in \mathrm{Cl}_{(X,d)}(A)$, and let (a_n) be a sequence in A that d-converges to x in X. By Lemma 1.2, (a_n) is a d-Cauchy sequence in X, and so it is a d_A-Cauchy sequence in A. But (A, d_A) is complete, and so (a_n) converges to a point in A. Since limits are unique in a metric space, this limit is x. Thus $x \in A$ so $\mathrm{Cl}_{(X,d)}(A) \subseteq A$.

See Book D, Chapter 14, Theorem 3.3.

2. *Let (X, d) be complete and let $A \subseteq X$ be d-closed. We show that (A, d_A) is complete.*

   If (a_n) is a d_A-Cauchy sequence in A, then it is also a d-Cauchy sequence in X. Since X is complete, (a_n) d-converges to some $x \in X$. Since each term of the sequence (a_n) is in A, the d-limit x belongs to $\mathrm{Cl}_{(X,d)}(A)$. Since A is d-closed, $A = \mathrm{Cl}_{(X,d)}(A)$ and so $x \in A$, meaning that (a_n) is d_A-convergent. Thus (A, d_A) is complete.

■

It follows immediately from Proposition 2.2 that, if $A \subseteq \mathbb{R}$, then $(A, d_A^{(1)})$ is a complete metric space if, and only if, A is $d^{(1)}$-closed.

Exercise 2.3

For which of the following sets A is $(A, d_A^{(1)})$ complete?

(a) $A = [0, 1]$

(b) $A = [0, 1)$

(c) $A = (0, 1)$

We now look at the relationship between completeness and compactness. Recall from Theorem 5.12 of Chapter 22 that metric spaces are compact if, and only if, they are sequentially compact. Since compact subsets of a metric space are closed, Proposition 2.2 also implies that compact subsets of complete metric spaces are complete. In fact, we can say more:

See Corollary 5.13 part 1 of Chapter 22.

Theorem 2.3

Each compact metric space is complete.

Proof Let (X, d) be a compact metric space and suppose that (a_n) is a Cauchy sequence in X. Since X is compact, then it is sequentially compact and hence (a_n) has a d-convergent subsequence, and so we conclude from Lemma 1.6 that (a_n) is d-convergent, as required. ■

Exercise 2.4

Let (X, d) be a compact metric space, let (Y, e) be a metric space, and let $f\colon X \to Y$ be a (d, e)-continuous onto function.

(a) Is (Y, e) complete?

(b) What can you deduce if f is not onto?

We now ask you to show that a finite metric space is complete.

Exercise 2.5

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite set, and let d be a metric on X. Prove that (X, d) is complete.

This is a particularly simple example of a complete metric space. Our remaining examples of complete metric spaces are more interesting.

2.2 The completeness of $(\mathbb{R}^n, d^{(n)})$

In Theorem 1.8 we showed that $(\mathbb{R}, d^{(1)})$ is a complete metric space. We now show that $(\mathbb{R}^n, d^{(n)})$ is a complete metric space, for any $n \geq 1$.

In Chapter 14, we saw that a sequence $(\mathbf{a}_k)$ in $\mathbb{R}^n$ converges to $\mathbf{a}$ with respect to the Euclidean metric $d^{(n)}$ if, and only if, each component sequence $(a_{j,k})_{k\in\mathbb{N}}$ converges to a_j, for $j = 1, 2, \ldots, n$. Using the same idea, we can show that $(\mathbb{R}^n, d^{(n)})$ is complete – that is, each $d^{(n)}$-Cauchy sequence in $\mathbb{R}^n$ converges to a point in $\mathbb{R}^n$.

See Book D, Chapter 14, Proposition 1.9.

Here $\mathbf{a} = (a_1, a_2, \ldots, a_n)$ and $\mathbf{a}_k = (a_{1,k}, a_{2,k}, \ldots, a_{n,k})$.

We first consider the relationship between a Cauchy sequence in $\mathbb{R}^n$ and the corresponding component sequences.

Lemma 2.4

A sequence $(\mathbf{a}_k)_{k\in\mathbb{N}}$ is a $d^{(n)}$-Cauchy sequence in $\mathbb{R}^n$ if, and only if, for each $j = 1, 2, \ldots, n$ the component sequence $(a_{j,k})_{k\in\mathbb{N}}$ is a $d^{(1)}$-Cauchy sequence in $\mathbb{R}$.

Proof Let $(\mathbf{a}_k)_{k\in\mathbb{N}}$ be a $d^{(n)}$-Cauchy sequence in $\mathbb{R}^n$. Given $\varepsilon > 0$, choose a $K \in \mathbb{N}$ such that

$$d^{(n)}(\mathbf{a}_k, \mathbf{a}_l) < \varepsilon \text{ for each } k, l > K.$$

For each $j \in \{1, 2, \ldots, n\}$,

$$|a_{j,k} - a_{j,l}| \leq d^{(n)}(\mathbf{a}_k, \mathbf{a}_l) < \varepsilon,$$

and so for each $j = 1, 2, \ldots, n$, $(a_{j,k})_{k\in\mathbb{N}}$ is a $d^{(1)}$-Cauchy sequence.

Conversely, suppose that $(a_{j,k})_{k\in\mathbb{N}}$ is a $d^{(1)}$-Cauchy sequence in $\mathbb{R}$ for each $j = 1, 2, \ldots, n$. Given $\varepsilon > 0$ and $j \in \{1, 2, \ldots, n\}$, choose a $K_j \in \mathbb{N}$ such that

$$|a_{j,k} - a_{j,l}| < \tfrac{1}{\sqrt{n}}\varepsilon \text{ for each } k, l > K_j.$$

Then, for $k, l > \max\{K_1, K_2, \ldots, K_n\}$,

$$d^{(n)}(\mathbf{a}_k, \mathbf{a}_l)^2 = \sum_{j=1}^{n} |a_{j,k} - a_{j,l}|^2 < \sum_{j=1}^{n} \left(\tfrac{1}{\sqrt{n}}\varepsilon\right)^2 = \varepsilon^2$$

and so $d^{(n)}(\mathbf{a}_k, \mathbf{a}_l) < \varepsilon$. Thus $(\mathbf{a}_k)_{k\in\mathbb{N}}$ is a $d^{(n)}$-Cauchy sequence in $\mathbb{R}^n$. ■

This lemma allows us to show that $(\mathbb{R}^n, d^{(n)})$ is complete.

Corollary 2.5

The metric space $(\mathbb{R}^n, d^{(n)})$ is complete.

Proof Let $(\mathbf{a}_k)_{k\in\mathbb{N}}$ be a $d^{(n)}$-Cauchy sequence in $\mathbb{R}^n$. By Lemma 2.4, $(a_{j,k})_{k\in\mathbb{N}}$ is a $d^{(1)}$-Cauchy sequence in $\mathbb{R}$, for $j = 1, 2, \ldots, n$, and so, since $\mathbb{R}$ is complete (see Theorem 1.8), it converges to $a_j \in \mathbb{R}$, say. Thus $(\mathbf{a}_k)$ is convergent (and converges to $\mathbf{a} = (a_1, a_2, \ldots, a_n)$). Hence $(\mathbb{R}^n, d^{(n)})$ is complete. ■

See Book D, Chapter 14, Proposition 1.9.

Exercise 2.6

Let $A = \{\mathbf{x} \in \mathbb{R}^n : d^{(n)}(\mathbf{x}, \mathbf{0}) \leq 1\}$. Show that $(A, d_A^{(n)})$ is a complete metric space.

2.3 Cantor space is complete

We now show that the Cantor space (with the Cantor metric) is a complete metric space. First let us recall the definition of Cantor space.

Cantor space (see Definitions 2.1 and 2.2 of Chapter 15)

The Cantor space $\mathbf{C}$ consists of all infinite sequences of zeros and ones:

$$\mathbf{C} = \{(a_n) : a_n \in \{0, 1\} \text{ for each } n \in \mathbb{N}\},$$

and the Cantor metric $d_{\mathbf{C}} : \mathbf{C} \times \mathbf{C} \to \mathbb{R}$ is given by

$$d_{\mathbf{C}}(\mathbf{x}, \mathbf{y}) = \begin{cases} 0, & \text{if } \mathbf{x} = \mathbf{y}, \\ 2^{-n}, & \text{if } \mathbf{x} \text{ and } \mathbf{y} \text{ first differ at their } n\text{th term.} \end{cases}$$

The fact that Cantor space is complete follows from the following observation concerning $d_{\mathbf{C}}$-Cauchy sequences.

Lemma 2.6

Let $(\mathbf{x}_n)$ be a $d_{\mathbf{C}}$-Cauchy sequence in $\mathbf{C}$. Then for each $k \in \mathbb{N}$, there is $N_k \in \mathbb{N}$ such that for $n, m > N_k$,

$$x_{i,n} = x_{i,m} \text{ for } i = 1, \ldots, k.$$

In other words, the initial entries of terms in a Cauchy sequence are eventually the same.

Exercise 2.7

Prove Lemma 2.6.

Theorem 2.7

$(\mathbf{C}, d_{\mathbf{C}})$ is a complete metric space.

Proof Let $(\mathbf{x}_n)$ be a $d_{\mathbf{C}}$-Cauchy sequence in $\mathbf{C}$. We must show that it is a $d_{\mathbf{C}}$-convergent sequence. To do this, we need to identify the limit of the sequence, $\mathbf{x}$ say, and then show that $\mathbf{x}_n \xrightarrow{d_{\mathbf{C}}} \mathbf{x}$ as $n \to \infty$.

Finding a candidate for the limit of $(\mathbf{x}_n)$

For $k \in \mathbb{N}$, let $N_k \in \mathbb{N}$ be as given in Lemma 2.6. We define the candidate limit sequence $\mathbf{x} = (x_1, x_2, x_3, \ldots)$ as follows:

$$x_k = x_{k,N_k+1} \text{ for each } k \in \mathbb{N}.$$

Showing that $\mathbf{x}_n \to \mathbf{x}$

Let $\varepsilon > 0$ and choose $k \in \mathbb{N}$ so that $2^{-k} < \varepsilon$, then for $n > N_k$, we have

$$x_{i,n} = x_i \text{ for each } i = 1, 2, \ldots, k$$

and so

$$d_{\mathbf{C}}(\mathbf{x}_n, \mathbf{x}) \leq 2^{-k} < \varepsilon.$$

Hence $(\mathbf{x}_n)$ converges to $\mathbf{x}$ for $d_{\mathbf{C}}$, as required. ■

Remark

Most of the effort here was concentrated on identifying the limit of the Cauchy sequence. Once a candidate limit had been identified, the verification that it was indeed the limit was straightforward.

2.4 $(C[0,1], d_{\max})$ is complete

We now turn to an important example of a complete metric space that is not a subset of a Euclidean space: $(C[0,1], d_{\max})$.

We have already studied this space in some detail. Recall that $d_{\max}$ is given by

$$d_{\max}(f, g) = \max\{|g(x) - f(x)| : x \in [0,1]\},$$

(Figure 2.1) and measures the maximum difference between the values of two continuous functions on $[0,1]$.

In this subsection, we prove the following result.

Theorem 2.8

The metric space $(C[0,1], d_{\max})$ is complete.

See Book D, Chapter 15, Section 5.

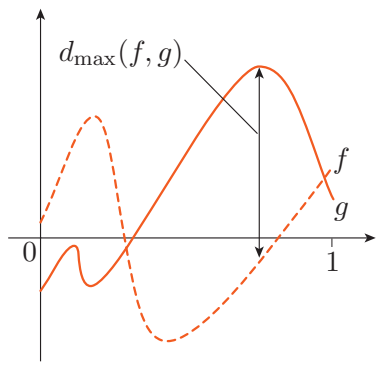


Figure 2.1

The proof of this result is sophisticated, and in order to prove it we use the notion of uniform convergence that we introduced in Section 4 of Chapter 15.

Recall that uniform convergence for a sequence of functions from $[0,1]$ to $\mathbb{R}$ is defined as follows.

Uniform convergence (see Definition 4.4 of Chapter 15)

A sequence (f_n) of functions $f_n\colon [0,1] \to \mathbb{R}$ **converges uniformly** on $[0,1]$ to the function $f\colon [0,1] \to \mathbb{R}$ if

1. the function $f_n - f$ is bounded for each $n \in N$, and
2. the sequence (M_n) defined by
$$M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$$
is a (real) null sequence.

Proof of Theorem 2.8 Let (f_n) be a $d_{\max}$-Cauchy sequence of functions in $C[0,1]$. The proof consists of three steps:

1. identify a candidate limit function, f, for the sequence (f_n)
2. show that (f_n) converges uniformly to f
3. deduce that f is continuous and that $d_{\max}(f_n, f) \to 0$ as $n \to \infty$.

If we do this, then we can conclude that (f_n) is $d_{\max}$-convergent to f in $C[0,1]$ and so $(C[0,1], d_{\max})$ is complete.

Identify a candidate limit function, f, of the sequence (f_n)

For each $x \in [0,1]$ and for each $n, m \in \mathbb{N}$,

$$|f_n(x) - f_m(x)| = d^{(1)}(f_n(x), f_m(x)) \leq d_{\max}(f_n, f_m).$$

Thus, for each $x \in [0,1]$, $(f_n(x))$ is a $d^{(1)}$-Cauchy sequence in $\mathbb{R}$. By Theorem 1.8, $(f_n(x))$ is $d^{(1)}$-convergent, and we define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \lim_{n\to\infty} f_n(x).$$

This function f is our candidate for the $d_{\max}$-limit of the sequence (f_n).

Show that (f_n) converges uniformly to f

We first show that the function $f_n - f$ is bounded for each $n \in \mathbb{N}$. To do this, we observe that Lemma 1.4 implies that there are an $M > 0$ and a $g \in C[0,1]$ such that

$$f_n \in B_{d_{\max}}(g, M) \text{ for each } n \in \mathbb{N}.$$

Hence for each $x \in [0,1]$ and each $n \in \mathbb{N}$,

$$|f_n(x) - g(x)| \leq d_{\max}(f_n, g) < M.$$

The Triangle Inequality now implies that, for each $x \in [0,1]$ and each $n \in \mathbb{N}$,

$$|f(x) - g(x)| \leq |f(x) - f_n(x)| + |f_n(x) - g(x)| < |f(x) - f_n(x)| + M.$$

For each $x \in [0,1]$, $|f(x) - f_n(x)| \to 0$ as $n \to \infty$, and so

$$|f(x) - g(x)| \leq M.$$

Using the Triangle Inequality again, we find that for each $x \in [0,1]$,

$$|f(x) - f_n(x)| \leq |f(x) - g(x)| + |g(x) - f_n(x)| \leq 2M.$$

Thus the function $f_n - f$ is bounded for each $n \in \mathbb{N}$.

We now show that

$$M_n = \sup\{|f_n(x) - f(x)| : x \in [0,1]\}$$

is a null sequence and so, by the definition of uniform convergence, (f_n) converges uniformly to f.

Let $\varepsilon > 0$. Since (f_n) is a $d_{\max}$-Cauchy sequence, there is an $N \in \mathbb{N}$ such that for each $m, n > N$,

$$d_{\max}(f_n, f_m) < \tfrac{1}{2}\varepsilon.$$

Let $x \in [0,1]$. Since $(f_n(x))$ converges to $f(x)$, there is an $M \in \mathbb{N}$ such that for each $m > M$,

$$|f(x) - f_m(x)| < \tfrac{1}{2}\varepsilon.$$

Note the introduction of the function f_m and the consequent disappearance of the dependence on x in the bound for $|f(x) - f_n(x)|$. This is a *uniformisation argument*: we find an estimate of $|f(x) - f_n(x)|$ that is independent of the choice of X. This is a useful technique.

Let $n > N$ and $m > \max\{M, N\}$. We use the Triangle Inequality to estimate that

$$\begin{aligned} |f(x) - f_n(x)| &\leq |f(x) - f_m(x)| + |f_m(x) - f_n(x)| \\ &\leq |f(x) - f_m(x)| + d_{\max}(f_n, f_m) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Since $x \in [0,1]$ is arbitrary, we conclude that for $n > N$,

$$M_n = \sup\{|f(x) - f_n(x)| : x \in [0,1]\} \leq \varepsilon.$$

But $\varepsilon > 0$ is arbitrary and so (M_n) is a null sequence. Thus (f_n) converges uniformly to f.

Deduce that $f \in C[0,1]$ *and that* $d_{\max}(f_n, f) \to 0$ *as* $n \to \infty$

Since (f_n) converges uniformly to f, the Uniform Convergence Theorem (Theorem 5.8 of Chapter 15) implies that f is continuous on $[0,1]$, and so is an element of $C[0,1]$, and $d_{\max}(f_n, f) \to 0$ as $n \to \infty$.

Hence (f_n) is $d_{\max}$-convergent to f in $C[0,1]$, as required. ■

In the next example, we give a metric on $C[0,1]$ for which the resulting space is *not* complete.

Example 2.9 $C[0,1]$ *with the integral metric*

The integral metric on $C[0,1]$ is given by

$$d(f,g) = \int_0^1 |g(x) - f(x)|\, dx.$$

We saw in Book D, Chapter 15, Worked Exercise 5.3 that this defines a metric on $C[0,1]$.

Define a sequence of functions (f_n) in $C[0,1]$ by setting, for $n \in \mathbb{N}$,

$$f_n(x) = \begin{cases} 0, & \text{if } 0 \le x \le \frac{1}{2}\left(1 - \frac{1}{n}\right), \\ 2n\left(x - \frac{1}{2}\left(1 - \frac{1}{n}\right)\right), & \text{if } \frac{1}{2}\left(1 - \frac{1}{n}\right) < x < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

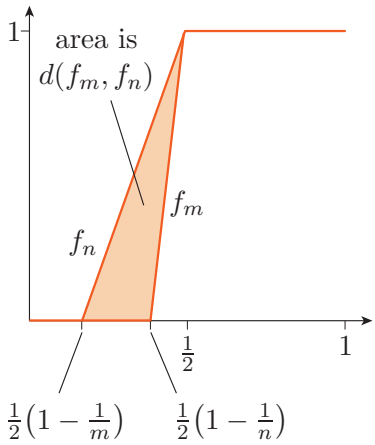


Figure 2.2

Either by integrating, or by finding the areas of appropriate triangles, we can easily show that if $N \in \mathbb{N}$ and $m > n > N$ (Figure 2.2), then

$$d(f_m, f_n) = \frac{1}{4}\left(\frac{1}{n} - \frac{1}{m}\right) < \frac{1}{2N}.$$

Hence (f_n) is a Cauchy sequence in $(C[0,1], d)$.

Now define $f\colon [0,1] \to \mathbb{R}$ by

$$f(x) = \begin{cases} 0, & \text{if } 0 \le x < \frac{1}{2}, \\ 1, & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$

Then,

$$\begin{aligned} \int_0^1 |f(x) - f_n(x)|\,dx &= \int_{\frac{1}{2}(1-\frac{1}{n})}^{\frac{1}{2}} 2n\left(x - \frac{1}{2}\left(1 - \frac{1}{n}\right)\right) dx \\ &= \left[n\left(x - \frac{1}{2}\left(1 - \frac{1}{n}\right)\right)^2\right]_{\frac{1}{2}(1-\frac{1}{n})}^{\frac{1}{2}} \\ &= n\left(\frac{1}{2n}\right)^2 = \frac{1}{4n}. \end{aligned}$$

Thus f is a candidate for the limit of the sequence (f_n). However, f is *not* an element of $C[0,1]$, since it is not continuous at the point $\frac{1}{2}$. In fact, for each $g \in C[0,1]$

$$\int_0^1 |g(x) - f(x)|\,dx > 0.$$

To see this, we first note that since g is continuous and f is not, we know that $f \neq g$. If f were continuous on $[0,1]$, we could then deduce that

$$d(f,g) = \int_0^1 |g(x) - f(x)|\,dx > 0.$$

Unfortunately, we cannot apply the metric d since f is not continuous and so not in $C[0,1]$. However, both f restricted to $[0, \frac{1}{2})$ and f restricted to $[\frac{1}{2}, 1]$ are continuous, and so we consider the metrics on $C[0, \frac{1}{2})$ and $C[\frac{1}{2}, 1]$ given respectively by

$$d_-(g,h) = \int_0^{\frac{1}{2}} |h(x) - g(x)|\,dx \text{ and } d_+(g,h) = \int_{\frac{1}{2}}^1 |h(x) - g(x)|\,dx.$$

The proof that d_- and d_+ are metrics is similar to the proof that d is a metric.

We know that there exists $x_0 \in [0,1]$ such that $f(x_0) \neq g(x_0)$ and so at least one of $d_-(f|_{[0,\frac{1}{2})}, g|_{[0,\frac{1}{2})})$ and $d_+(f|_{[\frac{1}{2},1]}, g|_{[\frac{1}{2},1]})$ is positive. Since

$$\int_0^1 |g(x) - f(x)|\,dx = d_-(f|_{[0,\frac{1}{2})}, g|_{[0,\frac{1}{2})}) + d_+(f|_{[\frac{1}{2},1]}, g|_{[\frac{1}{2},1]}),$$

we have $\int_0^1 |g(x) - f(x)| > 0$, as claimed.

We now use the Reverse Triangle Inequality to estimate

$$\begin{aligned} d(f_n, g) &= \int_0^1 |g(x) - f_n(x)|\, dx \\ &\geq \left| \int_0^1 |g(x) - f(x)|\, dx - \int_0^1 |f(x) - f_n(x)|\, dx \right| \\ &\geq \int_0^1 |g(x) - f(x)|\, dx - \int_0^1 |f(x) - f_n(x)|\, dx \\ &= \int_0^1 |g(x) - f(x)|\, dx - \frac{1}{4n}. \end{aligned}$$

But if we choose n large enough so that $\frac{1}{4n} < \frac{1}{2}\int_0^1 |g(x) - f(x)|\, dx$, then

$$d(f_n, g) > \tfrac{1}{2} \int_0^1 |g(x) - f(x)|\, dx > 0,$$

and so (f_n) does not converge to g.

Since g was an arbitrary function in $C[0,1]$, we conclude that the sequence (f_n) is not convergent and $(C[0,1], d)$ is not a complete metric space.

This is a little unfortunate in that there are many occasions when it is more natural to use the integral metric d rather than the $d_{\max}$ metric. This example shows that, in this case, it is not going to be enough to show that a sequence is Cauchy in order to deduce it is convergent.

This raises the question of whether we can find a complete metric space (X, d_X) that contains $(C[0,1], d)$ as a subspace. In fact, we can. In Section 4, we discuss a method for constructing the *smallest* complete space containing a given metric space as a subspace.

2.5 Completeness is not a homeomorphism invariant

Recall that in Chapter 21, we introduced the notion of homeomorphisms for metric spaces and saw that the property of being connected is a homeomorphism invariant: if (X, d) is connected and homeomorphic to (Y, e), then (Y, e) is connected. The following example shows that completeness is not a homeomorphism invariant.

Example 2.10

Let $\mathbb{R}$ have the Euclidean metric and let $X = (-1, 1)$ have the metric induced from the Euclidean metric for $\mathbb{R}$. The function $f\colon X \to \mathbb{R}$ given by $f(x) = \tan\left(\pi\frac{x}{2}\right)$ is a homeomorphism between $(-1, 1)$ and $\mathbb{R}$ (Figure 2.3 overleaf), and so $(X, d_X^{(1)})$ is homeomorphic to $(\mathbb{R}, d^{(1)})$.

We omit the proof that f is $(d_X^{(1)}, d^{(1)})$-continuous and f^{-1} is $(d^{(1)}, d_X^{(1)})$-continuous.

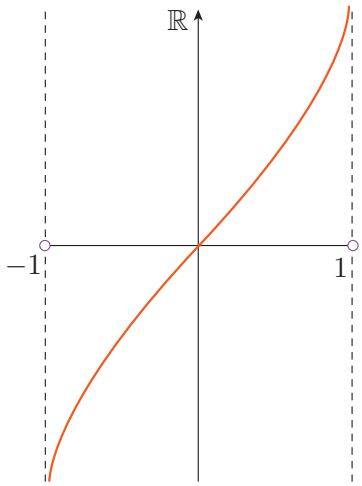


Figure 2.3

Now consider the sequence $\left(1-\frac{1}{n}\right)$ in X. It is a Cauchy sequence since, if $N \in \mathbb{N}$ and $n > m > N$, then

$$\left(1-\frac{1}{n}\right)-\left(1-\frac{1}{m}\right)=\frac{1}{m}-\frac{1}{n}<\frac{1}{N}\to\ 0 \text{ as } m,n\to\infty.$$

This sequence converges to 1 in $\mathbb{R}$, but does not converge in X, since $1 \notin (-1,1)$. Hence, $(X, d_X^{(1)})$ is incomplete.

But $(\mathbb{R}, d^{(1)})$ is complete and so we have two homeomorphic metric spaces, one complete and the other not: thus completeness is not a topological invariant.

We gain some insight into this example by noting that if we apply the homeomorphism f to the terms of the sequence $(1-\frac{1}{n})$ in $(-1,1)$, we obtain the image sequence $(\tan(\frac{\pi}{2}-\frac{\pi}{2n}))$ in $\mathbb{R}$. This is not a Cauchy sequence in $\mathbb{R}$; if it were, it would converge (since $\mathbb{R}$ is complete) and its limit would be $\tan\frac{\pi}{2}$, which is undefined.

Although completeness is not a homeomorphism invariant, it is a metric invariant:

if two metric spaces are metrically the same, then they are either both complete or both incomplete.

In order to make sense of this statement, we need to understand what it means for two metric spaces to be 'metrically the same'. The key property is the notion of distance: two metric spaces are metrically the same if we can map the points of one space to the other space, with a one–one and onto map, so that all distances are preserved. The following definition makes this precise.

Definition 2.11 *Isometric metric spaces*

Let (X, d_X) and (Y, d_Y) be metric spaces.

A function $f\colon X \to Y$ is an **isometry** if

1. f is one–one and onto
2. for each $x_1, x_2 \in X$, $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$.

The metric spaces (X, d_X) and (Y, d_Y) are **isometric** if there is an isometry between X and Y.

Remarks

1. If $f\colon X \to Y$ is an isometry, then it is invertible (since it is one–one and onto) and its inverse, f^{-1}, is also an isometry.
2. Since an isometry preserves distances and is a bijection, a sequence in X converges if, and only if, its image under the isometry also converges in Y. Equivalently, a sequence in X is a d_X-Cauchy sequence if, and only if, its image under the isometry is a d_Y-Cauchy sequence in Y.

See Exercise 2.9.

3. Since closed sets and sequentially compact sets can be characterised in terms of sequences, an isometry preserves closed and (sequentially) compact sets, and thus also preserves open sets. This implies that connectedness is also preserved under isometry.

Exercise 2.8

Let d be the Euclidean metric for $\mathbb{R}$. Show that:

(a) $([0,1], d_{[0,1]})$ is isometric to $([1,2], d_{[1,2]})$

(b) $([0,1], d_{[0,1]})$ is not isometric to $([0,2], d_{[0,2]})$.

Exercise 2.9

Let (X, d_X) and (Y, d_Y) be metric spaces, and let $f\colon X \to Y$ be an isometry. Show that if (a_n) is a sequence in X that is d_X-convergent to a, then $(f(a_n))$ is d_Y-convergent to $f(a)$ in Y.

Exercise 2.8(b) gives an example that shows that homeomorphic metric spaces need not be isometric. Thus the following theorem does not contradict Example 2.10.

Theorem 2.12

Let (X, d_X) and (Y, d_Y) be isometric metric spaces. If (X, d_X) is complete, then (Y, d_Y) is also complete.

Proof Suppose that (X, d_X) and (Y, d_Y) are isometric metric spaces and let $f\colon X \to Y$ be an isometry. Let (X, d_X) be complete, and let (a_n) be a d_Y-Cauchy sequence in Y. Then $(f^{-1}(a_n))$ is a d_X-Cauchy sequence in X since, for any $n, m \in \mathbb{N}$,

Since f is an isometry, it is invertible.

$$d_X(f^{-1}(a_n), f^{-1}(a_m)) = d_Y(a_n, a_m).$$

Since (X, d_X) is complete, there is an $x \in X$ such that $(f^{-1}(a_n))$ is d_X-convergent to x. But

$$d_Y(a_n, f(x)) = d_X(f^{-1}(a_n), x) \to 0, \text{ as } n \to \infty,$$

so (a_n) is d_Y-convergent to $f(x)$ in Y. Hence, (Y, d_Y) is complete. ■

Isometries play a big role in Section 4, where we discuss how to form the *completion* of a metric space – the smallest complete metric space that contains a given space.

3 The Contraction Mapping Theorem

In this section, we show how complete metric spaces play an important role in the theory of fixed points of mappings. Finding the fixed points of a particular function – that is, those points that are mapped to themselves – is frequently an important part of solving a mathematical problem.

We give a formal definition of fixed points in Definition 3.1.

3.1 Fixed points and the Contraction Mapping Theorem

Many problems in mathematics can be written in the form:

solve the equation $f(x) = x$.

Such points X are known as *fixed points* of f.

Definition 3.1 *Fixed point*

Let X be a set and let $T\colon X \to X$ be a function.

A **fixed point** of T is a point $x \in X$ such that $T(x) = x$.

We use T to denote the function and not f because the set X may have functions as its elements.

We shall see some examples shortly.

Fixed points are surprisingly important and there are many theorems that provide conditions that guarantee a function T has a fixed point. In this section, we are going to prove one such result, the Contraction Mapping Theorem. It ensures that, under appropriate conditions, a function has a *unique* fixed point. This theorem also provides a method for finding an approximate value for the fixed point.

Exercise 3.1

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and define $g\colon \mathbb{R} \to \mathbb{R}$ by $g(x) = x - f(x)$. Show that X is a fixed point of g if, and only if, X is a zero of f – that is, $f(X) = 0$.

Exercise 3.2

Let $f\colon \mathbb{R} \to \mathbb{R}$ be a function and for $r \in \mathbb{R} - \{0\}$ define $g_r\colon \mathbb{R} \to \mathbb{R}$ by $g_r(x) = x - rf(x)$. Show that X is a fixed point of g_r if, and only if, X is a zero of f.

Thus the problem of finding the zeros of a real-valued function can be converted into a problem concerning the fixed points of a (different) real-valued function. For example, finding the zeros of

$\cos x - \exp x$, for $x \in \mathbb{R}$ (see Figure 3.1),

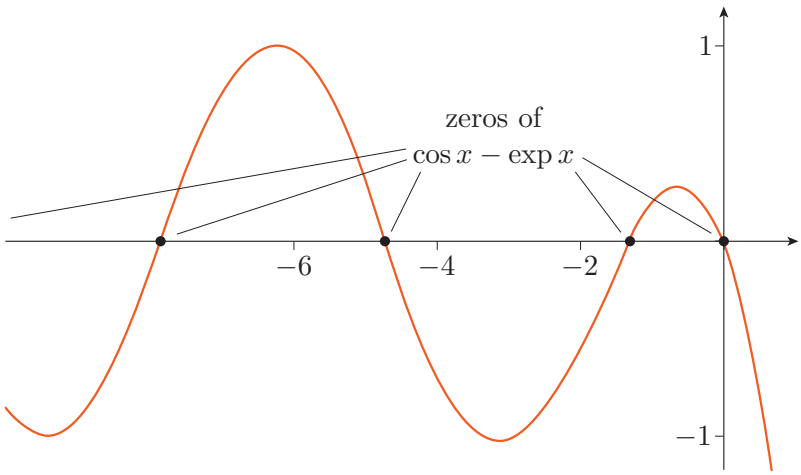


Figure 3.1

is the same as finding the fixed points of the function $g\colon \mathbb{R} \to \mathbb{R}$ given by

$$g(x) = x - \cos x + \exp x \quad \text{(see Figure 3.2).}$$

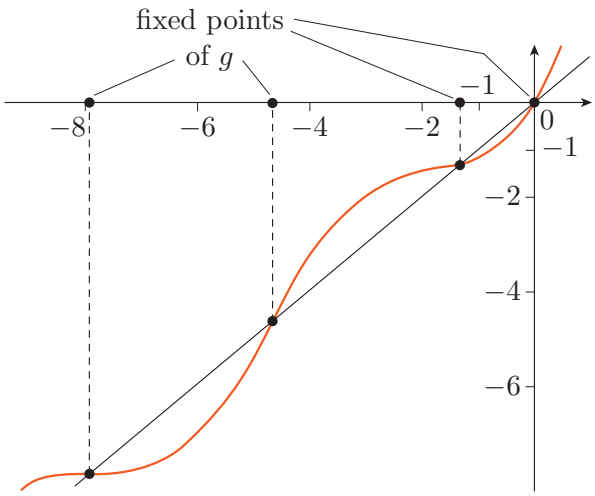


Figure 3.2

Our next example of a fixed-point problem involves a mapping defined on $C[0,1]$.

Example 3.2

Suppose that a function $f \in C[0,1]$ satisfies the differential equation

$$\frac{df}{ds} = sf(s), \quad f(0) = 1 \quad (0 \le s \le 1). \tag{1}$$

Integrating both sides of this equation with respect to s gives

$$\int_0^t \frac{df}{ds}\, ds = \int_0^t sf(s)\, ds$$

and so

$$f(t) - f(0) = \int_0^t sf(s)\,ds, \text{ for } 0 \le t \le 1.$$

Using the boundary condition $f(0) = 1$, we obtain an *integral equation* that any solution f of the differential equation must satisfy; namely,

$$f(t) = 1 + \int_0^t sf(s)\,ds, \text{ for } 0 \le t \le 1.$$

Let us define a map T on $C[0,1]$, by

$$T(f)(t) = 1 + \int_0^t sf(s)\,ds, \text{ for } f \in C[0,1]. \tag{2}$$

Then $T(f)$ is itself a function from $[0,1]$ to $\mathbb{R}$ and any solution f of the differential equation (1) is also a fixed point of the map T, $T(f) = f$.

We now show that T is a map from $C[0,1]$ to itself. First note that $s \mapsto sf(s)$ is continuous, since it is the product of continuous functions, and so belongs to $C[0,1]$. Also, the map F defined by

$$F(g)(t) = \int_0^t g(s)ds, \text{ for } 0 \le t \le 1,$$

See Book D, Chapter 15, Worked Exercise 5.7.

is a map from $C[0,1]$ to itself. So the function G defined by

$$G(f)(t) = \int_0^t sf(s)ds, \text{ for } 0 \le t \le 1,$$

is in $C[0,1]$. The constant function 1 is also in $C[0,1]$ and so T is a map from $C[0,1]$ to itself.

This example shows us that any solution of the differential equation (1) is a fixed point of the mapping T given in (2). Thus finding all the fixed points of the mapping T also gives us all the potential solutions to the differential equation. (It may be the case that the integral equation has fixed points that are not solutions of the differential equation – we will give criteria later that ensure this does not occur.)

Exercise 3.3

Consider the differential equation

$$\frac{df}{ds} = \tfrac{1}{2}\cos f(s), \quad f(0) = 0 \quad (0 \le s \le 1).$$

By integrating both sides of this equation, find an integral equation that every solution of this differential equation must satisfy.

Hence define a map $T\colon C[0,1] \to C[0,1]$ such that any solution of the differential equation is also a fixed point of this map.

The theorem we now prove is the Contraction Mapping Theorem. This theorem guarantees the existence of a fixed point when the mapping from a complete metric space to itself is a special type of map known as a *contraction mapping*.

Definition 3.3 *Contraction mapping*

Let (X, d) be a metric space. A function $T\colon X \to X$ is a **contraction mapping** for d if there is a real number $0 \leq \lambda < 1$ such that

$$d(T(x), T(y)) \leq \lambda d(x, y) \text{ for each } x, y \in X \quad \text{(Figure 3.3)}.$$

Any such real number λ is known as a **contraction ratio** for T.

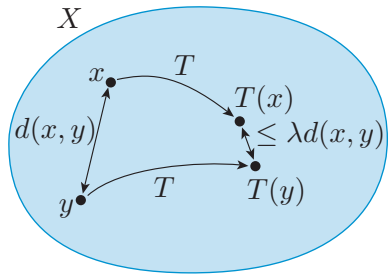


Figure 3.3

Remarks

1. We usually suppress mention of the metric and just say a map is a contraction mapping.
2. A contraction mapping is a Lipschitz function from a set X to itself with Lipschitz constant *strictly less than* 1. For example, $T\colon \mathbb{R} \to \mathbb{R}$ given by $T(x) = \frac{1}{2}x$ is a Lipschitz mapping with Lipschitz constant $\frac{1}{2}$ for the Euclidean metric on $\mathbb{R}$, and so is a contraction mapping.
3. A contraction mapping has many different possible contraction ratios: for if $\lambda \in (0, 1)$ is a contraction ratio, then so is any μ for which $\lambda < \mu < 1$.

Worked Exercise 3.4

Show that for the metric $d_{\max}$ on $C[0, 1]$, the map $T\colon C[0, 1] \to C[0, 1]$ defined by

$$T(f)(t) = 1 + \int_0^t s f(s) ds, \text{ for } 0 \leq t \leq 1$$

is a contraction mapping with contraction ratio $\frac{1}{2}$.

You met this map in Example 3.2.

Solution

We have already seen in Example 3.2 that T is a map from $C[0, 1]$ to $C[0, 1]$, and so it only remains to show that it is a contraction.

Given $f, g \in C[0, 1]$ and $t \in [0, 1]$, we have

$$\begin{aligned} |T(f)(t) - T(g)(t)| &= \left| \int_0^t s(f(s) - g(s))\, ds \right| \\ &\leq \int_0^t |s(f(s) - g(s))|\, ds \\ &\leq d_{\max}(f, g) \int_0^t s\, ds \\ &\leq d_{\max}(f, g) \int_0^1 s\, ds = \tfrac{1}{2} d_{\max}(f, g). \end{aligned}$$

Since t is arbitrary, we deduce that

$$d_{\max}(T(f), T(g)) \leq \tfrac{1}{2} d_{\max}(f, g),$$

and so T is a contraction mapping with contraction ratio $\frac{1}{2}$.

Exercise 3.4

Let (X, d) be a metric space and let $T\colon X \to X$ be a contraction mapping for d with contraction ratio 0. Show that T is a *constant* function.

Exercise 3.5

You met this map in Exercise 3.3.

Let $T\colon C[0,1] \to C[0,1]$ be defined by

$$T(f)(t) = \int_0^t \tfrac{1}{2} \cos f(s)\, ds, \text{ for } 0 \leq t \leq 1.$$

Show that T is a contraction mapping with contraction ratio $\frac{1}{2}$.

Hint: use the fact that $|\cos a - \cos b| \leq |a - b|$ for any real numbers a and b.

The next lemma shows that contraction mappings can have at most one fixed point. We will use this result later in the proof of the Contraction Mapping Theorem.

Lemma 3.5

Let (X, d) be a metric space. Suppose that $T\colon X \to X$ is a contraction mapping for d. Then T has at most one fixed point.

Proof Suppose that $x, y \in X$ are both fixed points of T – that is,

$$T(x) = x \text{ and } T(y) = y.$$

Since T is a contraction mapping,

$$d(x, y) = d(T(x), T(y)) \leq \lambda d(x, y),$$

where $0 \leq \lambda < 1$. Rearranging, we obtain

$$(1 - \lambda) d(x, y) \leq 0,$$

which implies that $d(x, y) \leq 0$, since $0 \leq \lambda < 1$. But $d(x, y) \geq 0$, and so $d(x, y) = 0$. Hence, since d is a metric, $x = y$. Thus T has at most one fixed point. ■

Remark

The conclusion of this lemma can fail if $T\colon X \to X$ is a Lipschitz map with Lipschitz constant 1 – that is if,

$$d(T(x), T(y)) \leq d(x, y) \text{ for } x, y \in X.$$

For example, consider $T\colon \mathbb{R} \to \mathbb{R}$ (with the Euclidean metric) where $T(x) = x$. For this mapping, *every* real number is a fixed point. This is why we require a contraction mapping to have a Lipschitz constant that is strictly less than one – it guarantees that it can have at most one fixed point.

Exercise 3.6

Let $(0, 1]$ have metric induced from the Euclidean metric for $\mathbb{R}$, and define $T\colon (0, 1] \to (0, 1]$ by $T(x) = \frac{1}{2}x$. Show that:

(a) T is a contraction mapping with contraction ratio $\frac{1}{2}$

(b) T has no fixed points.

Exercise 3.6 shows that a contraction mapping need not have any fixed points: the difficulty is caused by the fact that the point that 'should be' the fixed point of the mapping is not in the space. This difficulty will not arise for a *complete* metric space.

Theorem 3.6 *Contraction Mapping Theorem*

Let (X, d) be a complete metric space, and let $T\colon X \to X$ be a contraction mapping for d. Then there is a unique element $x_T \in X$ such that

$$T(x_T) = x_T.$$

Moreover, for each $x \in X$, the sequence of iterates

$$x,\ T(x),\ T(T(x)),\ T(T(T(x))),\ \ldots$$

converges to this unique fixed point x_T; that is, if

$$T^n(x) = (T \circ T \circ \cdots \circ T)(x) \ (T \text{ composed with itself } n \text{ times}),$$

then for each $x \in X$

$$d(T^n(x), x_T) \to 0 \quad \text{as } n \to \infty.$$

This is also known as the Banach Fixed Point Theorem after the Polish mathematician Stefan Banach (1892–1945). Banach was a prolific analyst and has many concepts and theorems named after him.

Proof Our proof involves three steps:

1. We show that for each $x \in X$, the sequence $x, T(x), T^2(x), \ldots, T^n(x), \ldots$ is a Cauchy sequence for d. Since (X, d) is complete, this sequence converges to some element $x_T \in X$.
2. We verify that $T(x_T) = x_T$ – that is, x_T is a fixed point of T.
3. We deduce from Lemma 3.5 that the fixed point is unique.

Since T is a contraction mapping for d, there is a number λ (with $0 \leq \lambda < 1$) such that

$$d(T(x), T(y)) \leq \lambda d(x, y) \text{ for each } x, y \in X.$$

Now fix $x \in X$ and consider the sequence of iterates: $x, T(x), T^2(x), \ldots$.

Show that $(T^n(x))$ is a Cauchy sequence

We need to show that for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$d(T^n(x), T^m(x)) < \varepsilon \text{ for each } n, m > N.$$

So let $\varepsilon > 0$ be given.

We start by estimating $d(T^k(x), T^{k+1}(x))$, the distance between two successive terms of the sequence, for $k = 1, 2, \ldots$:

$$\begin{aligned} d(T^k(x), T^{k+1}(x)) &= d(T(T^{k-1}(x)), T(T^k(x))) \\ &\leq \lambda d(T^{k-1}(x), T^k(x)), \text{ since } T \text{ is a contraction mapping} \\ &= \lambda d(T(T^{k-2}(x)), T(T^{k-1}(x))) \\ &\leq \lambda^2 d(T^{k-2}(x), T^{k-1}(x)), \text{ since } T \text{ is a contraction mapping} \\ &\vdots \quad \text{(repeating this } k \text{ times)} \\ &\leq \lambda^k d(x, T(x)). \end{aligned} \tag{3}$$

We define $T^0(x) = x$.

Let $n, m \in \mathbb{N}$ with $m > n$. Then, by (3) and repeated use of the Triangle Inequality for d, we find:

$$\begin{aligned} &d(T^n(x), T^m(x)) \\ &\leq d(T^n(x), T^{n+1}(x)) + d(T^{n+1}(x), T^{n+2}(x)) + \cdots + d(T^{m-1}(x), T^m(x)) \\ &\leq \lambda^n d(x, T(x)) + \lambda^{n+1} d(x, T(x)) + \lambda^{n+2} d(x, T(x)) + \cdots + \lambda^{m-1} d(x, T(x)) \\ &= (\lambda^n + \lambda^{n+1} + \lambda^{n+2} + \cdots + \lambda^{m-1}) d(x, T(x)) \\ &= \frac{\lambda^n - \lambda^m}{1 - \lambda} d(x, T(x)) \\ &\leq \frac{\lambda^n}{1 - \lambda} d(x, T(x)). \end{aligned} \tag{4}$$

Since $0 \leq \lambda < 1$ and x is fixed, we can choose $N \in \mathbb{N}$ so that

$$\frac{\lambda^N}{1 - \lambda} d(x, T(x)) < \varepsilon.$$

Hence for each $m > n > N$,

$$d(T^n(x), T^m(x)) \leq \frac{\lambda^n}{1 - \lambda} d(x, T(x)) \leq \frac{\lambda^N}{1 - \lambda} d(x, T(x)) < \varepsilon,$$

since $\lambda^n \leq \lambda^N$ for $0 \leq \lambda < 1$. Thus the sequence is a Cauchy sequence. But (X, d) is a *complete* metric space and so the sequence $(T^n(x))$ is d-convergent to some point $x_T \in X$.

Verify that x_T is a fixed point

By the Triangle Inequality for d, we have for each $n \geq 1$,

$$\begin{aligned} 0 \leq d(x_T, T(x_T)) &\leq d(x_T, T^n(x)) + d(T^n(x), T(x_T)) \\ &\leq d(x_T, T^n(x)) + \lambda d(x_T, T^{n-1}(x)). \end{aligned}$$

Since n is arbitrary and both $d(x_T, T^{n-1}(x))$ and $d(x_T, T^n(x)) \to 0$ as $n \to \infty$, we can take the limit as $n \to \infty$ on the right-hand side. We

conclude that

$0 \leq d(x_T, T(x_T)) \leq 0 + \lambda \cdot 0 = 0$

– that is, $d(x_T, T(x_T)) = 0$ and so $T(x_T) = x_T$.

Show that the fixed point is unique

This follows immediately from Lemma 3.5, which tells us that there is at most one fixed point. ■

Worked Exercise 3.7

Use the Contraction Mapping Theorem to show that the equation $x = \frac{1}{2}\cos x$ has a unique real solution.

We could also use the Intermediate Value Theorem to show that a root exists, but that would not show that there is exactly one root.

This is common in situations involving the Contraction Mapping Theorem.

Solution

This exercise does not specify which metric space we should work in. We choose a complete metric space and a contraction mapping T appropriate to the problem. In this case, there is a natural choice of mapping and space.

Let $T\colon \mathbb{R} \to \mathbb{R}$ be given by $T(x) = \frac{1}{2}\cos x$, and consider the complete metric space $(\mathbb{R}, d^{(1)})$.

Note that $T(x) = x$ if, and only if, $x = \frac{1}{2}\cos x$, and so the roots of this equation correspond exactly to the fixed points of the mapping T.

It remains only to check that T is a contraction. For $x, y \in \mathbb{R}$ with $x \neq y$, the Mean Value Theorem implies that there is $c \in (x, y)$ such that

$$\left|\frac{T(x) - T(y)}{x - y}\right| = \left|\frac{\frac{1}{2}(\cos x - \cos y)}{x - y}\right| = |T'(c)|.$$

See the *Handbook*.

Now

$$|T'(c)| = \left|\tfrac{1}{2}\sin c\right| \leq \tfrac{1}{2}, \text{ for } c \in \mathbb{R},$$

and so T is a contraction for the Euclidean metric with contraction ratio $\frac{1}{2}$. Hence we deduce from the Contraction Mapping Theorem that T has a unique fixed point, and so the original equation also has a unique real solution.

Exercise 3.7

Let $[1, \infty)$ have the induced Euclidean metric and define $T\colon [1, \infty) \to [1, \infty)$ by

$$T(x) = x + \tfrac{1}{x}.$$

(a) Show that

$$|T(x) - T(y)| < |x - y| \text{ for each } x, y \in [1, \infty).$$

(b) Does this map have a fixed point?

(c) Does this contradict the Contraction Mapping Theorem?

By looking at the proof of the Contraction Mapping Theorem, we can obtain an estimate of how far $T^n(x)$ is from x_T (the unique fixed point) in terms of $d(x, T(x))$. This is useful when using a computer to find an approximate value for the fixed point.

Corollary 3.8

Let (X, d) be a complete metric space and let $T\colon X \to X$ be a contraction mapping for d with contraction ratio $\lambda \in [0, 1)$. Then for each $x \in X$ and each $n \in \mathbb{N}$,

$$d(x_T, T^n(x)) \leq \frac{\lambda^n}{1-\lambda}\, d(x, T(x)),$$

where x_T is the fixed point of the map T.

Proof Let $x \in X$ and take $m, n \in \mathbb{N}$ with $m > n$. By the Triangle Inequality for d,

$$\begin{aligned} d(x_T, T^n(x)) &\leq d(x_T, T^m(x)) + d(T^m(x), T^n(x)) \\ &\leq d(x_T, T^m(x)) + \frac{\lambda^n}{1-\lambda}\, d(x, T(x)), \quad \text{by (4)}. \end{aligned}$$

But by the Contraction Mapping Theorem, $d(x_T, T^m(x)) \to 0$ as $m \to \infty$, and so, since m can be arbitrarily large, we deduce that

$$d(x_T, T^n(x)) \leq \frac{\lambda^n}{1-\lambda}\, d(x, T(x)).$$

■

Consider the mapping $T\colon \mathbb{R} \to \mathbb{R}$ given by $T(x) = \frac{1}{2}\cos x$. In Worked Exercise 3.7 we showed that T is a contraction with contraction ratio $\frac{1}{2}$ and so, by the Contraction Mapping Theorem, T has a unique fixed point x_T. We can now use Corollary 3.8 to estimate the value of x_T.

Taking $n = 1$ and $x = 0$, we have

$$|x_T - T(0)| \leq \frac{\frac{1}{2}}{1-\frac{1}{2}}|0 - T(0)| = \tfrac{1}{2}.$$

Since $T(0) = \frac{1}{2}$, the fixed point of T (which is the root of $x = \frac{1}{2}\cos x$) lies somewhere in $[0, 1]$. We can use Corollary 3.8 to obtain finer approximations to x_T by looking at $T^n(0)$ for larger values of n. For example, taking $n = 2$, we obtain

$$|x_T - T^2(0)| \leq \frac{(\frac{1}{2})^2}{1-\frac{1}{2}}|0 - T(0)| = \tfrac{1}{4}.$$

Since $T^2(0) = \frac{1}{2}\cos\frac{1}{2}$, this implies that x_T lies in the interval

$$[\tfrac{1}{2}\cos\tfrac{1}{2} - \tfrac{1}{4}, \tfrac{1}{2}\cos\tfrac{1}{2} + \tfrac{1}{4}] \subseteq [0.188, 0.689].$$

By increasing n, we can find smaller and smaller intervals that contain the fixed point.

Exercise 3.8

Consider the mapping $T\colon \mathbb{R} \to \mathbb{R}$ given by $T(x) = \frac{1}{2}\cos x$. Find an interval of length at most 0.1 containing the unique fixed point of T.

3.2 Applications of the Contraction Mapping Theorem

We now give some applications of the Contraction Mapping Theorem.

In the next chapter, we use the Contraction Mapping Theorem to define fractals.

Locating the zeros of real functions

If a continuous real-valued function on an interval is a contraction mapping, then we can use the Contraction Mapping Theorem to deduce the existence of a unique fixed point in that interval, and Corollary 3.8 to estimate how far we are from this fixed point when we iterate an initial guess.

Unfortunately, many of the functions whose zeros we wish to find cannot be directly rearranged to give a fixed point problem involving contractions. For example, if we wish to find the zeros of

$$x^2 - e^{-x}$$

(Figure 3.4) then, following Exercise 3.1, it is natural to look for the fixed points of the function $g\colon \mathbb{R} \to \mathbb{R}$ given by $g(x) = x - x^2 + e^{-x}$. But g is not a contraction mapping, and so we cannot apply the Contraction Mapping Theorem *directly*. However, by carefully defining an auxiliary function g, we *can* use the Contraction Mapping Theorem to find good approximations to the zeros of f.

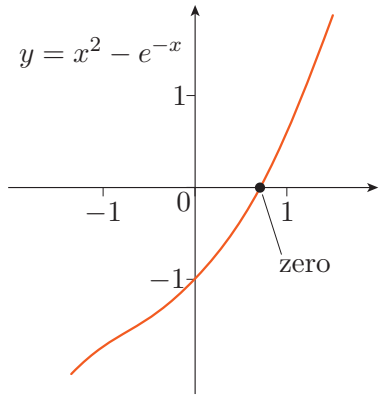


Figure 3.4

A sketch of the graph of f suggests that there is only one zero.

So suppose that we wish to find a zero of $f\colon \mathbb{R} \to \mathbb{R}$, given by $f(x) = x^2 - e^{-x}$. The function f is continuous, $f(0) = -1$ and $f(1) = 1 - e^{-1} > 0$. By the Intermediate Value Theorem, there is a zero of f in the interval $[0, 1]$.

The idea is to notice that the zeros of f correspond to the fixed points of the function g_r given by $g_r(x) = x - rf(x)$ (see Exercise 3.2) for *any* non-zero real number r. Thus, if we can find a non-zero real number r and a closed interval I such that $g_r\colon I \to I$ and g_r is a contraction mapping on this interval, then we can deduce from the Contraction Mapping Theorem (using the induced Euclidean metric for I) that f has a unique zero in this interval.

With this objective in mind, we estimate the derivative of g_r on the interval $[0, 1]$, where f has a zero. For f, we have $f'(x) = 2x + e^{-x}$, and so

$$1 \leq f'(x) \leq 3, \text{ for each } x \in [0, 1] \quad \text{(Figure 3.5 overleaf)}.$$

This tells us that f is strictly increasing on the interval $[0, 1]$.

This method tells us that there is a unique zero in the interval I but it does not tell us whether there are any zeros outside this interval. We need to use other methods to decide this.

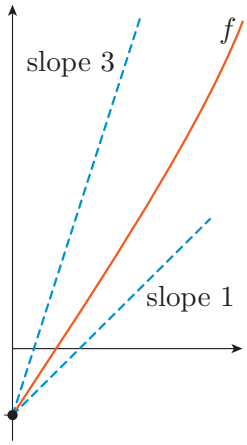


Figure 3.5

The function g_r has derivative

$$g_r'(x) = 1 - rf'(x), \text{ for all } x \in [0,1].$$

Hence for each $r > 0$,

$$1 - 3r \leq g_r'(x) \leq 1 - r \text{ whenever } 0 \leq x \leq 1.$$

Moreover, if $r \leq \frac{1}{3}$, then $1 - 3r \geq 0$ and so g_r is non-decreasing on $[0,1]$. Hence, for $0 \leq x \leq 1$ and $0 < r \leq \frac{1}{3}$,

$$g_r(x) \geq g_r(0) = 0 - rf(0) = r > 0$$

and

$$g_r(x) \leq g(1) = 1 - rf(1) = 1 - r\left(1 - \tfrac{1}{e}\right) < 1,$$

and so $g_r \colon [0,1] \to [0,1]$. Moreover, for $0 < r \leq \frac{1}{3}$, it follows from the Mean Value Theorem that for each $x, y \in [0,1]$,

$$|g_r(x) - g_r(y)| \leq \max_{0 \leq c \leq 1} |g_r'(c)||x - y| \leq (1 - r)|x - y|.$$

Thus, for $0 < r \leq \frac{1}{3}$, g_r is a contraction mapping on $[0,1]$ with contraction ratio $1 - r$. By the Contraction Mapping Theorem, g_r has a unique fixed point in the interval $[0,1]$ and thus f has a unique zero in $[0,1]$. If $r = \frac{1}{3}$ and x_r is the fixed point of g_r (and the zero of f) then, by Corollary 3.8,

$$|x_r - g_r^n(x_0)| \leq \frac{(1-r)^n}{1-(1-r)}|x_0 - g_r(x_0)| = 3\left(\tfrac{2}{3}\right)^n |x_0 - g_r(x_0)| \leq 3\left(\tfrac{2}{3}\right)^n,$$

whenever $x_0 \in [0,1]$ is an initial guess for the value of the fixed point. This method can be used to find a zero of any differentiable function f that is strictly monotonic on an open interval containing the zero being sought.

Theorem 3.9

Let $a < b \in \mathbb{R}$, and suppose that $f \colon [a,b] \to \mathbb{R}$ is continuous and differentiable on $[a,b]$. If

1. $f(a) < 0 < f(b)$
2. there are $m, M \in \mathbb{R}$ with $0 < m \leq M$ such that

$$m \leq f'(x) \leq M \text{ for each } x \in [a,b],$$

then for $0 < r \leq \frac{1}{M}$, the function $g_r \colon [a,b] \to \mathbb{R}$ given by $g_r(x) = x - rf(x)$ maps $[a,b]$ into itself and is a contraction mapping of $[a,b]$ with contraction ratio $1 - mr$.

It then follows from the Contraction Mapping Theorem (using the induced Euclidean metric for $[a,b]$) that a function f satisfying the conditions of Theorem 3.9 has a unique zero in $[a,b]$. We can then use Corollary 3.8 (applied to g_r) to find an approximate location for this zero to any desired accuracy.

Exercise 3.9

Prove Theorem 3.9.

Solving differential equations

Recall from Example 3.2 that any solution of the differential equation

$$\frac{df}{ds} = sf(s), \quad f(0) = 1 \quad (0 \leq s \leq 1)$$

must also satisfy the integral equation

$$f(t) = 1 + \int_0^t sf(s)\,ds \text{ for } 0 \leq t \leq 1.$$

We wish to find the fixed points of the map $T\colon C[0,1] \to C[0,1]$, given by

$$T(f)(t) = 1 + \int_0^t sf(s)\,ds.$$

In Worked Exercise 3.4, we saw that T is a contraction mapping for $d_{\max}$ with contraction ratio $\frac{1}{2}$. In Subsection 2.4, we proved that $(C[0,1], d_{\max})$ is a complete metric space. So, by the Contraction Mapping Theorem, there is a unique function $f \in C[0,1]$ for which $f = T(f)$ – that is,

$$f(t) = 1 + \int_0^t sf(s)\,ds. \tag{5}$$

And we can find good approximations to the fixed point of T by using Corollary 3.8.

We start by calculating $T(f_0)$ for a simple function $f_0 \in C[0,1]$ – say $f_0(t) = 1$ for each $t \in [0,1]$, so that

$$T(f_0)(t) = 1 + \int_0^t s\,ds = 1 + \tfrac{1}{2}t^2.$$

Hence,

$$|T(f_0)(t) - f_0(t)| = \tfrac{1}{2}t^2,$$

and so

$$d_{\max}(T(f_0), f_0) = \max\left\{\tfrac{1}{2}t^2 : t \in [0,1]\right\} = \tfrac{1}{2}.$$

Thus if $f \in C[0,1]$ is the fixed point of T, then by Corollary 3.8,

$$d_{\max}(f, T^n(f_0)) \leq \frac{(\frac{1}{2})^n}{1 - \frac{1}{2}} d_{\max}(T(f_0), f_0) \leq (\tfrac{1}{2})^n.$$

We now calculate $T^2(f_0)$ for $t \in [0,1]$: we have

$$\begin{aligned} T^2(f_0)(t) &= T(T(f_0))(t) \\ &= 1 + \int_0^t s(1 + \tfrac{1}{2}s^2)\,ds \\ &= 1 + \tfrac{1}{2}t^2 + \tfrac{1}{8}t^4 = 1 + \left(\tfrac{1}{2}t^2\right) + \tfrac{1}{2}\left(\tfrac{1}{2}t^2\right)^2. \end{aligned}$$

Similarly, we can find $T^3(f_0)$: for $t \in [0,1]$, we have

$$\begin{aligned} T^3(f_0)(t) &= 1 + \int_0^t s\left(1 + \left(\tfrac{1}{2}s^2\right) + \tfrac{1}{2}\left(\tfrac{1}{2}s^2\right)^2\right) ds \\ &= 1 + \left(\tfrac{1}{2}t^2\right) + \tfrac{1}{2!}\left(\tfrac{1}{2}t^2\right)^2 + \tfrac{1}{3!}\left(\tfrac{1}{2}t^2\right)^3. \end{aligned}$$

In fact, on noting that

$$\int_0^t s\left(\tfrac{1}{2}s^2\right)^k ds = \frac{t^{2(k+1)}}{2^k(2k+2)} = \frac{1}{k+1}\left(\tfrac{1}{2}t^2\right)^{k+1},$$

we can use mathematical induction to conclude that

$$T^n(f_0)(t) = \sum_{k=0}^{n} \frac{1}{k!}\left(\tfrac{1}{2}t^2\right)^k.$$

Since convergence for the $d_{\max}$ metric implies pointwise convergence, the fixed point f of T is given by

$$f(t) = \lim_{n\to\infty}\left(T^n f_0\right)(t) = \sum_{k=0}^{\infty} \frac{1}{k!}\left(\tfrac{1}{2}t^2\right)^k = \exp(\tfrac{1}{2}t^2).$$

Thus $f(t) = \exp(\frac{1}{2}t^2)$ is the fixed point of the map T.

Exercise 3.10

Let $f(t) = \exp(\frac{1}{2}t^2)$. Verify that f satisfies (5).

We have found the (unique) fixed point of T in $(C[0,1], d_{\max})$, but it still remains to verify that this function also satisfies our original differential equation

$$\frac{df}{ds} = sf(s), \quad f(0) = 1 \quad (0 \le s \le 1).$$

The following result, whose proof we omit, guarantees that a solution to the integral equation corresponding to a given differential equation is also a solution of the original differential equation.

Theorem 3.10

Suppose that $F\colon \mathbb{R}^2 \to \mathbb{R}$ is a $(d^{(2)}, d^{(1)})$-continuous function and $a \in \mathbb{R}$ is given. Then $f \in C[0,1]$ is a solution of the differential equation

$$\frac{df}{ds} = F(s, f(s)), \quad f(0) = a \quad (0 \le s \le 1)$$

if, and only if,

$$f(t) = a + \int_0^t F(s, f(s))\, ds \quad \text{for each } 0 \le t \le 1.$$

Exercise 3.11

(a) Use the results of Exercises 3.3 and 3.5 to show that the differential equation

$$\frac{df}{ds} = \tfrac{1}{2}\cos f(s), \quad f(0) = 0 \quad (0 \leq s \leq 1)$$

has a unique solution in $C[0,1]$.

(b) By taking $f_0(s) = 0$ for $0 \leq s \leq 1$ as your initial approximation to this solution f, find an estimate $g \in C[0,1]$ of f for which

$$d_{\max}(f,g) \leq \tfrac{1}{4}.$$

Hint: you will need to use Theorem 3.10 to show that you have a unique solution to the differential equation.

4 Completion

In Section 3 we saw that, in complete metric spaces, contraction mappings have a unique fixed point, and that this is useful for solving a range of mathematical problems. In this section we show you how to find the 'smallest' complete metric space that contains a given (presumably incomplete) metric space.

4.1 Defining the completion

Suppose that we are given a metric space (X,d) that is incomplete, that is it has at least one divergent Cauchy sequence. Is it always possible to find a complete metric space that contains (X,d) as a metric subspace? This is an important question, because if it is not possible, then the utility of the Contraction Mapping Theorem is restricted.

In Section 1, we saw that $((0,1], d^{(1)}_{(0,1]})$ is not a complete metric space, but that $(\mathbb{R}, d^{(1)})$ is complete and contains $((0,1], d^{(1)}_{(0,1]})$. But $([0,1], d^{(1)}_{[0,1]})$ also contains $((0,1], d^{(1)})$ as a metric subspace and $([0,1], d^{(1)}_{[0,1]})$ is a complete metric space by Proposition 2.2, since $[0,1]$ is a closed subset of $\mathbb{R}$. Thus we have several possible choices for a complete metric space containing $((0,1], d^{(1)})$, with $([0,1], d^{(1)}_{[0,1]})$ appearing to be the 'smallest'.

Given a metric space containing (X,d), we aim to identify the 'smallest' complete metric space that contains (X,d) as a subspace. We do this by looking for a complete metric space (X^*, d^*) for which (X,d) is a *dense* subspace – that is, $\mathrm{Cl}_{(X^*,d^*)}(X) = X^*$ and $d = d^*_X$ is the induced metric from d^* on X. In particular, each point $x \in X^*$ is a d^*-closure point of X and so $B_{d^*}(x,r) \cap X \neq \emptyset$ for each $r > 0$.

Strictly, we should indicate that these are metrics induced from the Euclidean metric for $\mathbb{R}$, but this makes the notation cumbersome and so we don't bother.

In the case of $((0,1], d^{(1)})$, it is straightforward to identify such a metric space (X^*, d^*): it is $([0,1], d^{(1)})$.

Exercise 4.1

Find a complete metric space (X^*, d^*) for which $(\mathbb{Q}, d^{(1)})$ is a dense subspace.

If we can find a complete metric space (Y, e) that contains (X, d) as a subspace (so $X \subseteq Y$ and d is the induced metric, e_X, on X), then we take $X^* = \mathrm{Cl}_{(Y,e)}(X)$ and let d^* be the induced metric (from e) on X^* – Proposition 2.2 then tells us that (X^*, d^*) is also a complete metric space. Moreover, by definition it is a complete metric space that contains (X, d) as a dense set and it is the *smallest* such space: for if $y \in \mathrm{Cl}_{(Y,e)}(X) - X = X^* - X$ is omitted, then there is a sequence in X that e-converges to y in Y (and hence is e-Cauchy) but does not converge in $X^* - \{y\}$.

A sequence (x_n), all of whose terms lie in X, is d-Cauchy if, and only if, it is e-Cauchy when considered as a sequence in Y.

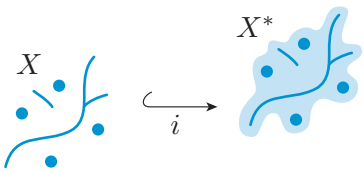
For each of the metric spaces $((0,1], d^{(1)})$ and $(\mathbb{Q}, d^{(1)})$, it is easy to find a complete metric space that contains it as a dense subspace, but in general it can be much harder to do this. For example, $(C[0,1], d)$ is not a complete metric space when d is the integral metric given by

$$d(f,g) = \int_0^1 |g(x) - f(x)|\, dx.$$

Example 2.9.

It seems difficult to identify a complete metric space that contains $(C[0,1], d)$. We seek a method that allows us to *construct* a complete metric space containing any given metric space as a dense subspace. This is not an easy thing to do, because we shall have to construct our space out of our (presumably incomplete) starting space.

In Section 2, we introduced the notion of *isometry* and observed that isometric metric spaces are metrically the same: any properties that depend only on the metric are common to both spaces. In particular, we saw in Theorem 2.12 that completeness is an *isometric invariant*: if (X, d_X) and (Y, d_Y) are isometric, then one is complete if, and only if, the other is. Hence it is enough for us to construct a complete metric space (X^*, d^*) that contains a dense subspace that is isometric to (X, d). Such a metric space (X^*, d^*) is called a *completion* of (X, d); see Figure 4.1.



$$X^* = \mathrm{Cl}_{(X^*,d^*)}(i(X))$$

Figure 4.1

Definition 4.1 *Completion of a metric space*

Let (X, d) be a metric space. A **completion** of X is a complete metric space (X^*, d^*) that contains a dense subspace isometric to (X, d).

Remarks

1. If (X^*, d^*) and (Y^*, e^*) are both completions of (X, d), then they are isometric – thus, up to isometry, completions are unique. Consequently, we usually talk about *the* completion of a space.

We shall not prove this.

2. Note that $([0,1], d^{(1)})$ is the completion of $((0,1], d^{(1)})$ and $(\mathbb{R}, d^{(1)})$ is the completion of $(\mathbb{Q}, d^{(1)})$.

Thus our task for the remainder of this chapter is to find a procedure that allows us to construct the completion of an arbitrary metric space. This is a conceptually difficult process and you should not be surprised if you find the remainder of this chapter heavy going. Your goal should be to understand the general approach and you should not be concerned if you find the details rather technical.

4.2 Constructing the completion of a space

Given a metric space (X, d), we aim to find a complete metric space (X^*, d^*) that contains a dense subspace isometric to (X, d). The key is to find a way of constructing a new metric space out of X that is 'bigger' than X and includes the limits of every Cauchy sequence in X.

The fundamental idea is to use the Cauchy sequences in X as 'points' in the new space X^*, so that the Cauchy sequence $(a, a, a, \ldots)$ represents the point $a \in X$, and a Cauchy sequence that does not converge in X is the point in X^* that will represent the limit of the sequence; for example, $(1, 1.4, 1.41, 1.414, 1.4142, \ldots)$, which is a Cauchy sequence in $(\mathbb{Q}, d^{(1)})$ that does not converge in $\mathbb{Q}$, should represent the point $\sqrt{2}$ in the completion.

Thus let X_C denote the collection of all the d-Cauchy sequences in X. So $\mathbf{a} = (a_n) = (a_1, a_2, a_3, a_4, \ldots) \in X_C$ if, and only if, for each $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that

$$d(a_n, a_m) < \varepsilon \text{ for each } n, m > N.$$

Exercise 4.2

When $(X, d) = (\mathbb{Q}, d^{(1)})$, which of the following are points in X_C?

(a) $(0, 0, 0, 0, \ldots)$ (b) $(1/n)_{n\in\mathbb{N}}$ (c) $(n)_{n\in\mathbb{N}}$

Unfortunately, the idea of defining the completion to be the set of all Cauchy sequences in X (with an appropriate metric) does not quite work. The problem is that two distinct Cauchy sequences may have the same limit, and so they should represent the same point. For example, the sequences $(0, 0, 0, \ldots)$ and $(1/n)_{n\in\mathbb{N}}$ should both represent the point 0.

We can perhaps better understand the problem by drawing an analogy with the problem of identifying $\{(p, q) : p, q \in \mathbb{Z}, q > 0\}$ with the set of rational numbers. We would like to say that if $p, q \in \mathbb{Z}$ with $q > 0$, then the pair (p, q) represents the rational number p/q. However, this representation is not unique, since the pairs $(1, 2)$ and $(2, 4)$ both represent the rational number $\frac{1}{2}$. Instead, we define an equivalence relation on $\{(p, q) : p, q \in \mathbb{Z}, q \neq 0\}$ by defining $(p, q) \sim (p', q')$ if $pq' = p'q$, and we then represent a rational number by the corresponding equivalence class.

We do the same here, and partition X_C into disjoint equivalence classes with each class representing a different point of our completion. A class will consist of all the Cauchy sequences with the same 'limit' – or at least it would, if we had a limit to refer to! But the whole point of this construction is that we do not have *a priori* limits for all Cauchy sequences when (X, d) is incomplete.

So the next stage of the construction is to determine an equivalence relation on X_C that splits X_C into equivalence classes of Cauchy sequences that 'should' have the same limit. The following lemma helps us to do this.

Lemma 4.2

Let (X, d) be a metric space. If (a_n) and (b_n) are sequences in X that converge to the same limit, then $d(a_n, b_n) \to 0$ as $n \to \infty$.

Proof Suppose that (a_n) and (b_n) are sequences in X that converge to the same limit $a \in X$, say, and let $\varepsilon > 0$ be given. Then

$$\text{there is an } N \in \mathbb{N} \text{ such that } d(a_n, a) < \tfrac{1}{2}\varepsilon \text{ for each } n > N,$$

and

$$\text{there is an } M \in \mathbb{N} \text{ such that } d(b_n, a) < \tfrac{1}{2}\varepsilon \text{ for each } n > M.$$

Hence, whenever $n > \max\{N, M\}$, we have $d(a_n, a) < \frac{1}{2}\varepsilon$ and $d(b_n, a) < \frac{1}{2}\varepsilon$, and, by the Triangle Inequality for d,

$$d(a_n, b_n) \leq d(a_n, a) + d(a, b_n) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

Thus $d(a_n, b_n) \to 0$ as $n \to \infty$. ■

We use this lemma to define a relation on X_C: if $\mathbf{a} = (a_n) \in X_C$ and $\mathbf{b} = (b_n) \in X_C$, then we say that $\mathbf{a}$ is equivalent to $\mathbf{b}$ (denoted $\mathbf{a} \sim \mathbf{b}$) if, and only if, $d(a_n, b_n) \to 0$ as $n \to \infty$.

Exercise 4.3

Show that $\sim$ is an equivalence relation on X_C.

We now define X^* to be the collection of equivalence classes of $(X_C, \sim)$.

We still need to:

1. define a metric d^* on X^*
2. show that (X, d) is isometric to a dense subset of (X^*, d^*)
3. show that (X^*, d^*) is complete.

Defining a metric on X^*

Let $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ be a Cauchy sequence in X, and let $[\mathbf{a}]$ denote the equivalence class of $\mathbf{a}$. Thus,

$$[\mathbf{a}] = \{\mathbf{a}' = (a'_1, a'_2, a'_3, \ldots) \in X_C : d(a_n, a'_n) \to 0 \text{ as } n \to \infty\}.$$

Given two points $[\mathbf{a}], [\mathbf{b}] \in X^*$, we define our metric d^* as follows:

$$d^*([\mathbf{a}], [\mathbf{b}]) = \lim_{n\to\infty} d(a_n, b_n), \tag{1}$$

where $(a_n) \in [\mathbf{a}]$ and $(b_n) \in [\mathbf{b}]$.

First, we must verify that d^* is well defined – that is, the limit in (1) exists, and if $\mathbf{a}' \in [\mathbf{a}]$ and $\mathbf{b}' \in [\mathbf{b}]$ are different choices of sequences from the equivalence classes, then

$$\lim_{n\to\infty} d(a'_n, b'_n) = \lim_{n\to\infty} d(a_n, b_n).$$

For otherwise, d^* would depend on which representatives of the equivalence class are used to calculate the limit, and so would not be well defined.

Lemma 4.3

Let $\mathbf{a} = (a_n) \in X_C$ and $\mathbf{b} = (b_n) \in X_C$. Then

1. $\lim_{n\to\infty} d(a_n, b_n)$ exists
2. if $\mathbf{a}' = (a'_n) \in [\mathbf{a}]$, $\mathbf{b}' = (b'_n) \in [\mathbf{b}]$, then
$$\lim_{n\to\infty} d(a'_n, b'_n) = \lim_{n\to\infty} d(a_n, b_n).$$

Proof

1. To show that $\lim_{n\to\infty} d(a_n, b_n)$ exists, we prove that $(d(a_n, b_n))$ is a Cauchy sequence in $\mathbb{R}$ for the Euclidean metric. For $n, m \in \mathbb{N}$, we use the Triangle Inequality for d, followed by the Reverse Triangle Inequality, as follows:
$$\begin{aligned} 0 &\le |d(a_n, b_n) - d(a_m, b_m)| \\ &= |(d(a_n, b_n) - d(a_m, b_n)) + (d(a_m, b_n) - d(a_m, b_m))| \\ &\le |d(a_n, b_n) - d(a_m, b_n)| + |d(a_m, b_n) - d(a_m, b_m)| \\ &\le d(a_n, a_m) + d(b_n, b_m). \end{aligned}$$
But (a_n) and (b_n) are both Cauchy sequences, and so $(d(a_n, b_n))$ is also a Cauchy sequence. Since $(\mathbb{R}, d^{(1)})$ is complete, $(d(a_n, b_n))$ converges.
2. Now suppose that $\mathbf{a}' \in [\mathbf{a}]$ and $\mathbf{b}' \in [\mathbf{b}]$.

   To prove that d^* is defined independently of the chosen representatives of the equivalence class, we show that
   $$|d(a'_n, b'_n) - d(a_n, b_n)| \to 0 \text{ as } n \to \infty.$$
   Since $\mathbf{a}' \in [\mathbf{a}]$ and $\mathbf{b}' \in [\mathbf{b}]$, we have $d(a'_n, a_n) \to 0$ and $d(b'_n, b_n) \to 0$.

Thus we use the Triangle Inequality, followed by the Reverse Triangle Inequality to estimate

$$\begin{aligned}|d(a'_n, b'_n) - d(a_n, b_n)| &= |d(a'_n, b'_n) - d(a'_n, b_n) + d(a'_n, b_n) - d(a_n, b_n)| \\ &\leq |d(a'_n, b'_n) - d(a'_n, b_n)| + |d(a'_n, b_n) - d(a_n, b_n)| \\ &\leq d(b'_n, b_n) + d(a'_n, a_n).\end{aligned}$$

But $d(b'_n, b_n) \to 0$ and $d(a'_n, a_n) \to 0$ as $n \to \infty$. Hence $|d(a'_n, b'_n) - d(a_n, b_n)| \to 0$ as $n \to \infty$ and the result follows.

■

Now we know that d^* is well defined, we can show that it defines a metric on X^*.

Theorem 4.4

(X^*, d^*) is a metric space.

Exercise 4.4

Prove Theorem 4.4.

Show that (X, d) is isometric to a dense subset of (X^*, d^*)

We start by considering which elements of X^* can be identified with the points of X. If $x \in X$, then $\mathbf{x} = (x, x, x, \ldots) \in X_C$ is a Cauchy sequence in X that converges to x. This gives us an embedding $i\colon X \to X^*$ where

$$i(x) = [(x, x, x, \ldots)] = [\mathbf{x}].$$

Thus $i(x)$ is the equivalence class consisting of all Cauchy sequences (a_n) in X for which $d(a_n, x) \to 0$ as $n \to \infty$ – that is, $i(x)$ consists of all sequences that converge to x.

Clearly for each $x, y \in X$,

$$d^*(i(x), i(y)) = \lim_{n\to\infty} d(x, y) = d(x, y),$$

and so $i(x) = i(y)$ if, and only if, $d(x, y) = 0$ which (by (M1)) holds if, and only if, $x = y$. Hence i is one–one; this implies that $i(X) = \{i(x) : x \in X\}$ is an isometric copy of X in X^*, and so (X, d) is isometric to $(i(X), d^*_{i(X)})$.

In fact, $i(X)$ is d^*-dense in X^*, as we now prove.

Lemma 4.5

The set $i(X)$ is a d^*-dense subset of (X^*, d^*).

Proof It is enough to show that for each $a^* \in X^*$ and each $\varepsilon > 0$, there is an $x \in X$ with $d^*(a^*, i(x)) < \varepsilon$.

See Book D, Chapter 16, Theorem 3.8.

So let $a^* \in X^*$, and let $\varepsilon > 0$ be given.

Since $a^* \in X^*$, there is a Cauchy sequence $\mathbf{a} = (a_n)$ in X such that $a^* = [\mathbf{a}]$. Choose an $N \in \mathbb{N}$ such that $d(a_n, a_m) < \frac{1}{2}\varepsilon$ for each $m, n > N$. Then for $x = a_{N+1}$,

$$d^*(a^*, i(x)) = d^*(a^*, i(a_{N+1})) = \lim_{n\to\infty} d(a_n, a_{N+1}).$$

But for $n > N$, $d(a_n, a_{N+1}) < \frac{1}{2}\varepsilon$, and so

$$d^*(a^*, i(x)) \leq \tfrac{1}{2}\varepsilon < \varepsilon.$$

Thus $i(X)$ is d^*-dense in X^*. ■

Showing that (X^*, d^*) is complete

Theorem 4.6

The metric space (X^*, d^*) is complete.

Proof We need to show that each d^*-Cauchy sequence in X^* converges to an element of X^*.

So suppose that $(A_m)_{m\in\mathbb{N}}$ is a d^*-Cauchy sequence in X^*. Each term A_m of the sequence is an equivalence class of Cauchy sequences from X. Hence for each $m \in \mathbb{N}$, we can choose a representative of the equivalence class, $\mathbf{a}_m = (a_{1,m}, a_{2,m}, a_{3,m}, \ldots)$ say, so that $A_m = [\mathbf{a}_m]$. Moreover, for each $m \in \mathbb{N}$, $(a_{n,m})_{n\in\mathbb{N}}$ is itself a d-Cauchy sequence in X.

Thus $(A_m)_{m\in\mathbb{N}}$ is a Cauchy sequence consisting of sets of Cauchy sequences!

We first identify a candidate for the limit of the sequence $([\mathbf{a}_m])$. To do this, we use Lemma 4.5 to find for each $m \in \mathbb{N}$, a point $x_m \in X$ such that

$$d^*(A_m, i(x_m)) = d^*([\mathbf{a}_m], i(x_m)) < \frac{1}{m}.$$

If we show that

1. $(x_m)_{m\in\mathbb{N}}$ is a d-Cauchy sequence in X,
2. $d^*(A_m, [(x_1, x_2, x_3, \ldots)]) = d^*([\mathbf{a}_m], [(x_1, x_2, x_3, \ldots)]) \to 0$ as $m \to \infty$,

then we can deduce that (A_m) is d^*-convergent (to $[(x_1, x_2, x_3, \ldots)]$) in X^*, as required.

$(x_m)_{m\in\mathbb{N}}$ is a d-Cauchy sequence in X

Let $\varepsilon > 0$. For each $m, n \in \mathbb{N}$ and each $j \in \mathbb{N}$ we can use the Triangle Inequality for d to estimate

$$d(x_m, x_n) \leq d(x_m, a_{j,m}) + d(a_{j,m}, a_{j,n}) + d(a_{j,n}, x_n).$$

But

$$\lim_{j\to\infty} d(a_{j,m}, x_m) = d^*([\mathbf{a}_m], i(x_m)) = d^*(A_m, i(x_m)) < \frac{1}{m}$$

and

$$\lim_{j\to\infty} d(a_{j,n}, x_n) = d^*([\mathbf{a}_n], i(x_n)) = d^*(A_n, i(x_n)) < \frac{1}{n}.$$

Moreover, $(A_m) = ([\mathbf{a}_m])_{m\in\mathbb{N}}$ is a Cauchy sequence in X^*, and so there is an $N \in \mathbb{N}$ such that for each $n, m > N$,

$$d^*(A_m, A_n) = d^*([\mathbf{a}_m], [\mathbf{a}_n]) = \lim_{j\to\infty} d(a_{j,m}, a_{j,n}) < \tfrac{1}{3}\varepsilon.$$

So if we choose $n, m > \max\{N, 3/\varepsilon\}$, and choose $j \in \mathbb{N}$ so that $d(a_{j,m}, a_{j,n}) < \frac{1}{3}\varepsilon$, then

$$\begin{aligned} d(x_m, x_n) &\leq d(x_m, a_{j,m}) + d(a_{j,m}, a_{j,n}) + d(a_{j,n}, x_n) \\ &< \frac{1}{m} + \frac{1}{3}\varepsilon + \frac{1}{n} \\ &< \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon + \frac{1}{3}\varepsilon = \varepsilon. \end{aligned}$$

Hence, $(x_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in X.

$d^*(A_m, [(x_1, x_2, x_3, \ldots)]) = d^*([\mathbf{a}_m], [(x_1, x_2, x_3, \ldots)]) \to 0$ *as* $m \to \infty$

Let $\varepsilon > 0$ be given. Then for each $m \in \mathbb{N}$, we use the Triangle Inequality for d^* to estimate

$$\begin{aligned} d^*([\mathbf{a}_m], [(x_1, x_2, x_3, \ldots)]) &\leq d^*([\mathbf{a}_m], i(x_m)) + d^*(i(x_m), [(x_1, x_2, x_3, \ldots)]) \\ &< \frac{1}{m} + \lim_{j\to\infty} d(x_m, x_j). \end{aligned}$$

But $(x_j)_{j\in\mathbb{N}}$ is a d-Cauchy sequence, and so we can find an $N \in \mathbb{N}$ such that $d(x_m, x_j) < \frac{1}{2}\varepsilon$ for each $j, m > N$. Hence if we choose $m > \max\{N, 2/\varepsilon\}$, then

$$d^*([\mathbf{a}_m], [(x_1, x_2, x_3, \ldots)]) < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

and so $A_m = [\mathbf{a}_m] \xrightarrow{d^*} [(x_1, x_2, x_3, \ldots)]$ as $m \to \infty$, as required. ■

Our results are summarised in the following theorem.

Theorem 4.7

Any metric space (X, d) has a completion (X^*, d^*).

We have shown that any metric space can be embedded as a dense subset of some complete metric space. However, the abstract construction that we have given is of limited utility in practice – it is useful for showing that the completion exists, but is hard to understand and make use of for concrete problems. It does not give us a clear understanding of the structure of the completion for any particular space: this must be obtained by special methods. In fact, only a few completions are known in detail, such as $\mathbb{Q}^* = \mathbb{R}$. However, if such a representation is found, it is unique up to isometry.

We omit the proof of this.

Returning briefly to $(C[0,1], d)$, where d is the integral metric, Theorem 4.7 tells us that there is a complete metric space (X^*, d^*) that contains $(C[0,1], d)$ as a dense subspace. In fact, the completion of $(C[0,1], d)$ is the space known as $L^1[0,1]$, the set of *Lebesgue integrable functions* on the interval $[0,1]$, and the metric is based on an extension of the definition of the integral for continuous functions known as the Lebesgue integral. This space plays a fundamental role in the theory of modern analysis, but study of it is beyond the scope of this course.

$d(f,g) = \int_0^1 |f(x) - g(x)|\, dx.$

Remark

The Lebesgue integral is a generalisation of the Riemann integral – it agrees with the Riemann integral for continuous functions, but is also defined for many functions that are not continuous, such as the function $f\colon [0,1] \to \mathbb{R}$ given by $f(x) = 1$ if X is rational, and 0 otherwise – the Lebesgue integral of f is 0.

Learning outcomes

We list those topics on which we may set assessment questions to test your understanding of the chapter.

After working through this chapter you should be able to:

(a) Test whether given sequences in a metric space are Cauchy.

(b) Determine when given subsets of standard metric spaces are complete.

(c) Recognise standard complete metric spaces.

(d) Use the Contraction Mapping Theorem to determine whether certain maps have a unique fixed point, and to find approximations to the fixed point that are within a specified distance.

(e) Understand the approach to constructing the completion of a given metric space.

Solutions to exercises

Solution to Exercise 1.1

(a) If (o_n) is the odd subsequence, then

$$\begin{aligned} o_n = b_{2n-1} &= -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^{2n-1}}{2n-1} \\ &= -1 + \frac{1}{2} - \frac{1}{3} + \cdots - \frac{1}{2n-1} \end{aligned}$$

for $n \in \mathbb{N}$ and so

$$o_{n+1} - o_n = -\frac{1}{2n+1} + \frac{1}{2n} = \frac{1}{4n^2+2n} > 0 \text{ for } n \in \mathbb{N}.$$

Hence (o_n) is monotonic increasing. Moreover, for each $n \in \mathbb{N}$,

$$o_n = \left(-1 + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{4}\right) + \cdots + \frac{-1}{2n-1} < 0$$

and so, by the Monotone Convergence Theorem (o_n) is a convergent sequence.

Similarly, if (e_n) is the even subsequence, then

$$\begin{aligned} e_n = b_{2n} &= -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^{2n}}{2n} \\ &= -1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{1}{2n} \end{aligned}$$

for $n \in \mathbb{N}$ and so

$$\begin{aligned} e_{n+1} - e_n &= \frac{-1}{2n+1} + \frac{1}{2n+2} \\ &= \frac{-1}{2(n+1)(2n+1)} < 0 \text{ for } n \in \mathbb{N}. \end{aligned}$$

Hence (e_n) is monotonic decreasing. Moreover, for each $n \in \mathbb{N}$,

$$\begin{aligned} e_n &= \left(-1 + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{-1}{2n-1} + \frac{1}{2n}\right) \\ &= -\frac{1}{2} - \frac{1}{12} - \cdots - \frac{1}{2n(2n-1)} \geq -\frac{1}{1} - \frac{1}{3^2} - \cdots - \frac{1}{4n^2} \\ &\geq -a_{2n} \geq -2 \end{aligned}$$

and so, by the Monotone Convergence Theorem (e_n) is a convergent sequence.

(b) For each $n \in \mathbb{N}$, we find

$$\begin{aligned} d_n &= \left(-1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^{2n-1}}{2n-1}\right) - \left(-1 + \frac{1}{2} - \frac{1}{3} + \cdots + \frac{(-1)^{2n}}{2n}\right) \\ &= -\frac{1}{2n}, \end{aligned}$$

which is a null sequence.

(c) Since d_n is null, it follows that both (o_n) and (e_n) converge to the same limit. In other words, both the odd and even subsequences of (b_n) converge to the same limit, which implies that (b_n) is also convergent (and has the same limit).

Solution to Exercise 1.2

(a) If $n, m > N$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} + \frac{1}{m} < \frac{2}{N}.$$

Now let $\varepsilon > 0$. Choose $N \in \mathbb{N}$ so that $N > \frac{2}{\varepsilon}$. If $n, m > N$, then

$$\left|\frac{1}{n} - \frac{1}{m}\right| < \frac{2}{N} < \varepsilon.$$

So $(1/n)$ is a Cauchy sequence.

Alternatively, $(1/n)$ is d-convergent with limit 0; so, by Lemma 1.2, it is a d-Cauchy sequence.

(b) Let $\varepsilon = 1$ and let $N \in \mathbb{N}$. Then

$$|(N+2) - (N+1)| = 1 = \varepsilon$$

and so $(n)_{n\in\mathbb{N}}$ is not a Cauchy sequence. (We could have used a similar argument for any value of $\varepsilon > 0$.)

Solution to Exercise 1.3

By Worked Exercise 3.5 of Chapter 14, eventually constant sequences are convergent; so, by Lemma 1.2, we deduce that they are also Cauchy.

Solution to Exercise 1.4

(a) To show that (c_n) is a Cauchy sequence, we note that

$$c_{n+1} - c_n = \frac{(-1)^{n+1}}{2^{n+1}}.$$

Hence if $m = n + i$ for some $i \in \mathbb{N}$, then

$$\begin{aligned} |c_{n+i} - c_n| &= |(c_{n+i} - c_{n+i-1}) + (c_{n+i-1} - c_{n+i-2}) + \cdots + (c_{n+1} - c_n)| \\ &\leq \frac{1}{2^{n+i}} + \frac{1}{2^{n+i-1}} + \cdots + \frac{1}{2^{n+1}} \\ &\leq \frac{1}{2^n}. \end{aligned}$$

Thus for each $\varepsilon > 0$, if we choose $N \in \mathbb{N}$ so that $2^{-N} < \varepsilon$, then for $n, m > N$, we have

$$|a_n - a_m| < \varepsilon$$

and so (c_n) is Cauchy as required.

(b) If (e_n) denotes the subsequence of even terms, so $e_n = c_{2n}$, then

$$e_{n+1} - e_n = -\frac{1}{2^{2n+2}} < 0,$$

and so (e_n) is a monotonic decreasing sequence. Moreover, $e_n \geq -\sum_{i=1}^{2n} \frac{1}{2^i} \geq -2$ and so it is bounded below. Hence by the Monotone Convergence Theorem (e_n) is a convergent sequence and we have found a convergent subsequence of (c_n).

We deduce from Lemma 1.6 that (c_n) is convergent.

Solution to Exercise 2.1

In Exercise 1.2 you saw that the sequence $(1/n)$ is a $d^{(1)}$-Cauchy sequence in $\mathbb{R}$. Since each term of the sequence lies in $(0, 1]$, it follows that the sequence is also Cauchy for the induced metric on $(0, 1]$. However, although each term of this sequence lies in $(0, 1]$, its limit 0 is not a member of $(0, 1]$ and so the sequence does not converge in $(0, 1]$. Thus, $(0, 1]$ with the induced Euclidean metric is incomplete.

Solution to Exercise 2.2

Certainly it is the case that convergent sequences are Cauchy (in any metric space) and so it only remains to show that Cauchy sequences are convergent. So suppose that (a_n) is a Cauchy sequence in (X, d_0). Then, in particular, there is $N \in \mathbb{N}$ such that for each $n, m > N$, $d_0(a_n, a_m) < 1$. But, since d_0 is the discrete metric for X, this means that for each $n, m > N$, $a_n = a_m$. In other words, the sequence (a_n) is *eventually constant*. But by Worked Exercise 3.5 of Chapter 14, eventually constant sequences are convergent (whatever the choice of metric) and so (X, d_0) is complete as required.

Solution to Exercise 2.3

The set $[0, 1]$ is the only one of the sets given that is closed for the Euclidean metric, and so it is the only set A for which $(A, d_A^{(1)})$ is complete. By considering the sequence $(1 - \frac{1}{n})$ for $[0, 1)$ and the sequence $(\frac{1}{n+1})$ for $(0, 1)$, it is possible to verify directly that neither of these sets is complete for the metric induced from the Euclidean metric.

Solution to Exercise 2.4

(a) The continuous image $f(X)$ of a compact space X is itself compact, by Corollary 5.13 of Chapter 22, and so, since $f(X) = Y$, (Y, e) is compact. Hence, by Theorem 2.3, it is complete.

(b) If f is not onto, then $f(X)$ is a compact *proper* subset of Y and hence complete but without further information, we cannot determine whether (Y, e) is itself complete.

Solution to Exercise 2.5

See Exercise 4.7 of Chapter 22.

Since finite sets are compact, (X, d) is a compact metric space and the result follows immediately from Theorem 2.3.

Solution to Exercise 2.6

We know that the closed unit ball $A = B_{d^{(n)}}[\mathbf{0}, 1]$ is $d^{(n)}$-closed (see Proposition 1.3 of Chapter 16) and that $(\mathbb{R}^n, d^{(n)})$ is complete (see Corollary 2.5). Hence by Proposition 2.2 we conclude that $(A, d_A^{(n)})$ is complete.

Solution to Exercise 2.7

Let $(\mathbf{x}_n)$ be a $d_{\mathbf{C}}$-Cauchy sequence in $\mathbf{C}$ and let $k \in \mathbb{N}$ be given.

Then for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that for $n, m > N$, $d_{\mathbf{C}}(\mathbf{x}_n, \mathbf{x}_m) < \varepsilon$. In particular, if we let $\varepsilon = 2^{-k}$, then there is $N_k \in \mathbb{N}$ such that for $n, m > N_k$, $d_{\mathbf{C}}(\mathbf{x}_n, \mathbf{x}_m) < 2^{-k}$. In particular, we must have $\mathbf{x}_n$ and $\mathbf{x}_m$ agreeing for their first k terms.

Solution to Exercise 2.8

(a) We must find an isometry from $[0, 1]$ to $[1, 2]$. Define $f\colon [0, 1] \to [1, 2]$ by $f(x) = x + 1$. It is easy to see that f is one–one and onto, and it remains only to verify that f preserves distances.

So let $x, y \in [0, 1]$; then

$$\begin{aligned} d_{[1,2]}(f(x), f(y)) &= |(x + 1) - (y + 1)| \\ &= |x - y| = d_{[0,1]}(x, y), \end{aligned}$$

and so distances are preserved by f.

Hence f is an isometry and $([0, 1], d_{[0,1]})$ and $([1, 2], d_{[1,2]})$ are isometric.

(b) To show that two spaces are not isometric, it is enough to show that there are points in one space that are at a distance not realised in the other space. In this case, the distance between the points $0, 2 \in [0, 2]$ is 2, whereas the distance between any two points in $[0, 1]$ is at most 1. Hence the two spaces are not isometric.

Solution to Exercise 2.9

Suppose that (a_n) is a sequence in X that is d_X-convergent to a. Then, since f is an isometry,

$$d_Y(f(a_n), f(a)) = d_X(a_n, a) \to 0 \text{ as } n \to \infty,$$

and so $(f(a_n))$ is a sequence in Y that is d_Y-convergent to $f(a)$.

Solution to Exercise 3.1

The point X is a fixed point of g if, and only if,

$$g(X) = X,$$

that is, if, and only if,

$$X - f(X) = X.$$

This is equivalent to $f(X) = 0$. Thus, X is a fixed point of g if, and only if, X is a zero of f.

Solution to Exercise 3.2

The point X is a fixed point of g_r if, and only if,

$$g_r(X) = X,$$

that is, if, and only if,

$$X - rf(X) = X.$$

Since $r \neq 0$, this is equivalent to $f(X) = 0$. Thus, X is a fixed point of g_r if, and only if, X is a zero of f.

Solution to Exercise 3.3

Integrating both sides of the differential equation with respect to s gives

$$\int_0^t \frac{df}{ds}\, ds = \int_0^t \tfrac{1}{2}\cos f(s)\, ds$$

and so

$$f(t) - f(0) = \int_0^t \tfrac{1}{2}\cos f(s)\, ds, \text{ for } 0 \leq t \leq 1.$$

Since $f(0) = 0$, this gives the integral equation

$$f(t) = \int_0^t \tfrac{1}{2}\cos f(s)\, ds, \text{ for } 0 \leq t \leq 1.$$

Thus, if we define a map T by

$$T(f)(t) = \int_0^t \tfrac{1}{2}\cos f(s)\, ds, \text{ for } 0 \leq t \leq 1,$$

then any solution of the differential equation is also a fixed point of this map T.

We now show that T is a map from $C[0,1]$ to itself. Since $s \mapsto (\frac{1}{2}\cos \circ f)(s)$ is a composite of continuous functions, it is continuous. Also, the map F defined by

$$F(g)(t) = \int_0^t g(s)\, ds, \text{ for } 0 \leq t \leq 1,$$

is a map from $C[0,1]$ to itself (as shown in Book D, Chapter 15, Worked Exercise 5.7) and so T is a map from $C[0,1]$ to itself. All solutions of the differential equation correspond to fixed points of the integral equation $T(f) = f$.

Solution to Exercise 3.4

Let $a \in X$ and let $c = T(a)$. We show that $T(x) = c$, for each $x \in X$.

Let us estimate $d(T(x), T(a))$. Since T has contraction ratio 0,

$$0 \leq d(T(x), c) = d(T(x), T(a)) \leq 0 \times d(x, a) = 0,$$

for each $x \in X$. Hence $d(T(x), c) = 0$ and so, since d is a metric, $T(x) = c$. In other words, T is a constant function.

Solution to Exercise 3.5

For $f, g \in C[0,1]$ and $t \in [0,1]$, we have

$$\begin{aligned}
|T(f)(t) - T(g)(t)| &= \left| \int_0^t \tfrac{1}{2}(\cos f(s) - \cos g(s))\, ds \right| \\
&\leq \tfrac{1}{2} \int_0^t |\cos f(s) - \cos g(s)|\, ds \\
&\leq \tfrac{1}{2} \int_0^t |f(s) - g(s)|\, ds, \text{ using the hint} \\
&\leq \tfrac{1}{2} \int_0^t d_{\max}(f,g)\, ds \\
&= \tfrac{1}{2} t d_{\max}(f,g) \leq \tfrac{1}{2} d_{\max}(f,g), \text{ since } 0 \leq t \leq 1.
\end{aligned}$$

Hence,

$$d_{\max}(T(f), T(g)) \leq \tfrac{1}{2} d_{\max}(f,g),$$

and so T is a contraction mapping for $d_{\max}$ with contraction ratio $\frac{1}{2}$.

Solution to Exercise 3.6

(a) It is enough to show that T is a Lipschitz mapping on $(0,1]$ with Lipschitz constant $\frac{1}{2}$.

If $x, y \in (0,1]$, then

$$|T(x) - T(y)| = |\tfrac{1}{2}x - \tfrac{1}{2}y| = \tfrac{1}{2}|x - y|.$$

Hence T is a Lipschitz mapping on $(0,1]$ with Lipschitz constant $\frac{1}{2}$.

(b) If $T(x) = x$, then $\frac{1}{2}x = x$; that is, $\frac{1}{2}x = 0$. So the only possible fixed point of T is the point 0, which does not belong to the domain of T, $(0,1]$. Thus T has no fixed points.

Solution to Exercise 3.7

(a) For each $x, y \in [1, \infty)$,

$$\begin{aligned}
|T(x) - T(y)| &= \left| \left(x + \frac{1}{x}\right) - \left(y + \frac{1}{y}\right) \right| \\
&= \left| x - y + \frac{y - x}{xy} \right| \\
&= \left| (x - y)\left(1 - \frac{1}{xy}\right) \right| \\
&= |x - y| \times \left| 1 - \frac{1}{xy} \right| \\
&< |x - y|.
\end{aligned}$$

(b) If T has a fixed point X, then $x = T(x)$. This means that $x = x + 1/x$, and so $1/x = 0$, which is impossible. Thus T has no fixed points.

(c) This does not contradict the Contraction Mapping Theorem, because T is not a contraction mapping with contraction ratio strictly less than 1.

Solution to Exercise 3.8

By Corollary 3.8,

$$|x_T - T^n(0)| \leq \frac{\left(\frac{1}{2}\right)^n}{1-\frac{1}{2}}|0 - T(0)| = \left(\frac{1}{2}\right)^n.$$

This places x_T in an interval of length $(1/2)^{n-1}$. Now

$$((1/2)^{n-1}) = (1, \tfrac{1}{2}, \tfrac{1}{4}, \tfrac{1}{8}, \tfrac{1}{16}, \tfrac{1}{32}, \cdots).$$

The first term in this sequence which is less than 0.1 is the term corresponding to $n = 5$. We have

$$|x_T - T^5(0)| \leq \left(\frac{1}{2}\right)^5 = \frac{1}{32}.$$

Since $T^5(0) \in (0.4502, 0.4503)$ and $\frac{1}{32} = 0.03125$,

$$x_T \in [T^5(0) - \tfrac{1}{32}, T^5(0) + \tfrac{1}{32}] \subseteq [0.41895, 0.48155] \subseteq (0.41, 0.49).$$

This places x_T in an interval of length less than 0.1.

Solution to Exercise 3.9

We follow the method used in the text before the theorem. Since $0 < m \leq f'(x) \leq M$ on $[a, b]$, we find for each $x \in [a, b]$,

$$1 - Mr \leq g_r'(x) = 1 - rf'(x) \leq 1 - mr.$$

Hence, if $0 < r \leq \frac{1}{M}$, then $1 - Mr \geq 0$, and so g_r is non-decreasing on $[a, b]$. In this case, for $x \in [a, b]$,

$$g_r(x) \geq g_r(a) = a - rf(a) > a$$

and

$$g_r(x) \leq g_r(b) = b - rf(b) < b.$$

Hence, for $0 < r \leq \frac{1}{M}$, g_r maps $[a, b]$ into itself. Finally, we apply the Mean Value Theorem to g_r to conclude that if $0 < r \leq \frac{1}{M}$, then for $x, y \in [a, b]$,

$$\begin{aligned} |g(x) - g(y)| &\leq \max_{c \in [a,b]} |g_r'(c)||x - y| \\ &\leq (1 - mr)|x - y|, \end{aligned}$$

and so g_r is a contraction mapping (for the Euclidean metric) on $[a, b]$ with contraction ratio $1 - mr$.

Solution to Exercise 3.10

For each $t \in [0,1]$,

$$\begin{aligned} T(f)(t) &= 1 + \int_0^t sf(s)\,ds \\ &= 1 + \int_0^t se^{\frac{1}{2}s^2}\,ds \\ &= 1 + \left[e^{\frac{1}{2}s^2}\right]_0^t \\ &= 1 + \left(e^{\frac{1}{2}t^2} - 1\right) = f(t). \end{aligned}$$

Hence $T(f) = f$, as required.

Solution to Exercise 3.11

(a) By the result of Exercise 3.3, any solution to the differential equation must also be a fixed point of the mapping $T\colon C[0,1] \to C[0,1]$, given by

$$T(f)(t) = \int_0^t \tfrac{1}{2}\cos f(s)\,ds \text{ for each } 0 \leq t \leq 1.$$

In Exercise 3.5, we saw that this is a contraction mapping (for $d_{\max}$) with contraction ratio $\frac{1}{2}$. Hence by the Contraction Mapping Theorem, there is a unique function $f \in C[0,1]$ that is a fixed point of T. Since $(s,y) \mapsto \frac{1}{2}\cos y$ is a $(d^{(2)}, d^{(1)})$-continuous function, it follows from Theorem 3.10 that f is also the unique solution of the original differential equation.

(b) Taking $f_0(s) = 0$ as our initial guess for the fixed point f, we find that

$$T(f_0) = \tfrac{1}{2}t$$

and

$$T(T(f_0)) = \int_0^t \tfrac{1}{2}\cos\tfrac{1}{2}s\,ds = \sin\tfrac{1}{2}t.$$

We deduce from Corollary 3.8 that

$$\begin{aligned} d_{\max}(f, \sin\tfrac{1}{2}t) &\leq \frac{(\frac{1}{2}^2)}{1-\frac{1}{2}} d_{\max}(f_0, T(f_0)) \\ &= \frac{\frac{1}{4}}{1-\frac{1}{2}} \times \tfrac{1}{2} = \tfrac{1}{4}, \end{aligned}$$

and so we can take $g(t) = \sin\frac{1}{2}t$ as our estimate to the solution of the differential equation.

(In fact, according to Wolfram Alpha, the solution of this differential equation is given by $f(s) = 2\tan^{-1}\left(\tanh\left(\frac{s}{4}\right)\right)$.)

Solution to Exercise 4.1

In Worked Exercise 3.9 of Chapter 16, we saw that $\mathbb{Q}$ is a dense subset of $\mathbb{R}$ for the Euclidean metric, and so, since Theorem 1.8 tells us that $(\mathbb{R}, d^{(1)})$ is a complete metric space, we take $(X^*, d^*) = (\mathbb{R}, d^{(1)})$.

Solution to Exercise 4.2

Both $(0, 0, 0, 0, \ldots)$ and $(1/n)$ are Cauchy sequences of rational numbers, and so are points in X_C.

However, $(n)_{n\in\mathbb{N}}$ is not a Cauchy sequence, and so is not in X_C.

Solution to Exercise 4.3

We need to show that $\sim$ is reflexive, symmetric and transitive on X_C.

Reflexive: $\mathbf{a} \sim \mathbf{a}$. Let $\mathbf{a} = (a_1, a_2, a_3, \ldots) \in X_C$. For each $n \in \mathbb{N}$, $d(a_n, a_n) = 0$, and so $d(a_n, a_n) \to 0$. Hence $\mathbf{a} \sim \mathbf{a}$.

Symmetric: $\mathbf{a} \sim \mathbf{b}$ *if, and only if,* $\mathbf{b} \sim \mathbf{a}$. Let $\mathbf{a} = (a_1, a_2, a_3, \ldots)$ and $\mathbf{b} = (b_1, b_2, b_3, \ldots)$ belong to X_C. Since $d(a_n, b_n) = d(b_n, a_n)$ for each $n \in \mathbb{N}$, we deduce that $d(a_n, b_n) \to 0$ if, and only if, $d(b_n, a_n) \to 0$ as $n \to \infty$. Thus, $\mathbf{a} \sim \mathbf{b}$ if, and only if, $\mathbf{b} \sim \mathbf{a}$.

Transitive: if $\mathbf{a} \sim \mathbf{b}$ *and* $\mathbf{b} \sim \mathbf{c}$ *then* $\mathbf{a} \sim \mathbf{c}$. Let $\mathbf{a} = (a_1, a_2, a_3, \ldots), \mathbf{b} = (b_1, b_2, b_3, \ldots)$ and $\mathbf{c} = (c_1, c_2, c_3, \ldots)$ belong to X_C.

If $\mathbf{a} \sim \mathbf{b}$ and $\mathbf{b} \sim \mathbf{c}$, then, for each $n \in \mathbb{N}$, the Triangle Inequality for d implies that

$$0 \leq d(a_n, c_n) \leq d(a_n, b_n) + d(b_n, c_n).$$

But $d(a_n, b_n) \to 0$ and $d(b_n, c_n) \to 0$ as $n \to \infty$, and so $d(a_n, c_n) \to 0$ as $n \to \infty$. Hence, $\mathbf{a} \sim \mathbf{c}$.

Thus $\sim$ is an equivalence relation on X_C.

Solution to Exercise 4.4

We show that d^* defines a metric on X^*. In Lemma 4.3 we showed that $d^* : X^* \times X^* \to \mathbb{R}$. We now need to verify properties (M1)–(M3). Throughout the following we let $\mathbf{a} = (a_n)$, $\mathbf{b} = (b_n)$ and $\mathbf{c} = (c_n)$.

(M1) By definition, $d^*([\mathbf{a}], [\mathbf{b}]) \geq 0$, and

$$d^*([\mathbf{a}], [\mathbf{a}]) = \lim_{n\to\infty} d(a_n, a_n) = 0.$$

If $d^*([\mathbf{a}], [\mathbf{b}]) = 0$, then $d(a_n, b_n) \to 0$; this implies that $\mathbf{a} \sim \mathbf{b}$, and so $[\mathbf{a}] = [\mathbf{b}]$. Hence (M1) holds.

(M2) Since d is a metric we have, for $[\mathbf{a}], [\mathbf{b}] \in X^*$,

$$\begin{aligned} d^*([\mathbf{a}], [\mathbf{b}]) &= \lim_{n\to\infty} d(a_n, b_n) \\ &= \lim_{n\to\infty} d(b_n, a_n) = d^*([\mathbf{b}], [\mathbf{a}]), \end{aligned}$$

and so (M2) holds.

(M3) Let $[\mathbf{a}], [\mathbf{b}], [\mathbf{c}] \in X^*$. Since the Triangle Inequality for d implies

$$d(a_n, c_n) \leq d(a_n, b_n) + d(b_n, c_n),$$

and Lemma 4.3 implies that $\lim_{n\to\infty} d(a_n, c_n)$, $\lim_{n\to\infty} d(a_n, b_n)$ and $\lim_{n\to\infty} d(b_n, c_n)$ exist, it follows that

$$\lim_{n\to\infty} d(a_n, c_n) \leq \lim_{n\to\infty} d(a_n, b_n) + \lim_{n\to\infty} d(b_n, c_n).$$

Hence,

$$d^*([\mathbf{a}], [\mathbf{c}]) \leq d^*([\mathbf{a}], [\mathbf{b}]) + d^*([\mathbf{b}], [\mathbf{c}]),$$

as required.

Since (M1)–(M3) are satisfied, d^* is a metric on X^*.